Core 4 June 2006 – Miss Watson's rough solutions

(1) $8x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$

$$8x + 2y + \frac{dy}{dx}(2x + 2y) = 0$$

$$\frac{dy}{dx} = \frac{-8x - 2y}{2x + 2y} \qquad (1,2)$$

$$\frac{dy}{dx} = \frac{-8 - 4}{2 + 4}$$

$$= \frac{-12}{6}$$

$$= -2$$

$\boxed{4}$

(2) (i) $(1 - 3x)^{-2}$

$x = (-3x)$

$n = -2$

$$1 + (-2)(-3x) + \frac{-2(-3)}{1.2}(-3x)^2$$

$$1 + 6x + 27x^2$$

$\boxed{3}$

(ii) $(1 + 2x)^2$

	1	$+2x$
1	1	$+2x$
$+2x$	$+2x$	$+4x^2$

$1 + 4x + 4x^2$

	1	$+4x$	$+4x^2$
1	1	$+4x$	$+4x^2$
$+6x$	$+6x$	$+24x^2$	$+24x^3$
$+27x^2$	$+27x^2$	$+108x^3$	$+108x^4$

$1 + 10x + 55x^2 \ldots$ 55

$\boxed{4}$

(3) (i) $\frac{3-2x}{x(3-x)}$

$\cancel{\frac{A(3-x)}{x}}$ $\frac{A}{x} + \frac{B}{(3-x)}$

$A(3-x) + Bx = 3 - 2x$
$3A - Ax + Bx = 3 - 2x$

$-Ax + Bx = -2x$
$3A = 3$
$\underline{\underline{A = 1}}$

$-1x + Bx = -2x$
$B = -1$

$\frac{1}{x} - \frac{1}{3-x}$ $\boxed{3}$

(ii) $\int_1^2 \frac{1}{x} - \frac{1}{3-x} \, dx$

$\left[\ln x + \ln(3-x)\right]_1^2$

$(\ln 2 + \ln 1) - (\ln 1 + \ln 2)$

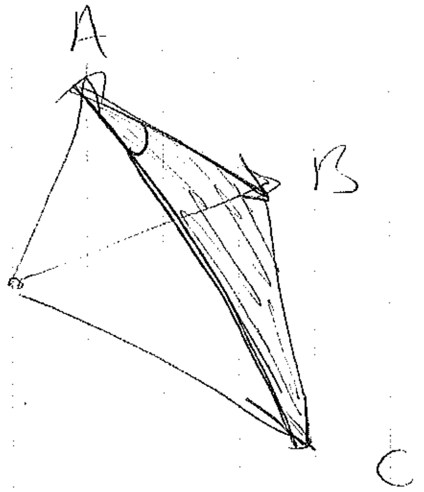
$\ln 2 - \ln 2 = \underline{\underline{0}}$ $\boxed{4}$

(iii) Area above and below axis are equal.

$\boxed{1}$

4

(i)



$$A = \begin{pmatrix} 7 \\ 3 \\ -3 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix} \quad C = \begin{pmatrix} 5 \\ 4 \\ -5 \end{pmatrix}$$

$$\overrightarrow{AB} = -B + A = \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = -A + C \quad \begin{pmatrix} -7 \\ -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$$

$$\cos\theta = \frac{a.b}{|a||b|}$$

$$\begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = 6 + {}^{-}1 + 2 = \underline{7}$$

$$|a| = \sqrt{9+1+1} = \sqrt{11} = \cancel{2\sqrt{3}}$$
$$|b| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\cos\theta = \frac{7}{3\sqrt{11}}$$

$$\theta = \cos^{-1}\left(\frac{7}{3\sqrt{11}}\right) = \cancel{47.66^\circ} = \underline{45.29^\circ}$$

$\boxed{6}$

(ii)

$$A = \frac{1}{2} ab \sin C$$
$$A = \frac{1}{2}\sqrt{11} \times 3 \times \sin(45.29)$$
$$\underline{A = 3.54}$$

$\boxed{2}$

5 (i) $\frac{dA}{dt} = kA^2$ $\boxed{2}$

(ii) $\frac{dt}{dA} = \frac{1}{kA^2}$

$$\int \frac{dt}{dA}\, dA = \int \frac{1}{k} A^{-2}\, dA$$

$$t = -\frac{1}{k} A^{-1} + c$$

$1 = \frac{-1}{1000k} + c$ —— $c = 1 + \frac{1}{1000k}$

$2 = \frac{-1}{2000k} + c$

$2 = \frac{-1}{2000k} + 1 + \frac{1}{1000k}$

$1 = \frac{-1}{2000k} + \frac{1}{1000k}$ $\times k$

$k = -\frac{1}{2000} + \frac{1}{1000} = \frac{1}{2000}$

$c = 1 + \frac{1}{1000 \times \frac{1}{2000}} = 1 + \frac{1}{0.5} = 1 + 2 = 3$ $\boxed{6}$

$t = \frac{-2000}{3000} + 3 = \frac{-2}{3} + 3 = 2\frac{1}{3} = \frac{7}{3}$ hours

6 (i) $u = e^x + 1$

$$\int \frac{e^{2x}}{e^x+1}\,dx$$

$$\int \frac{e^{2x}}{u}\,dx$$

$$u = e^x + 1$$

$$\frac{du}{dx} = e^x$$

$$\boxed{dx = \frac{1}{e^x}\,du}$$

$$\int \frac{e^{2x}}{u} \times \frac{1}{e^x}\,du$$

$$\int \frac{e^x}{u}\,du$$

$$\boxed{e^x = u \cancel{+} - 1}$$

$$\int \frac{u \cancel{+} - 1}{u}\,du$$

$\boxed{3}$

(ii)

$$\int_{\cancel{1}\,2}^{\cancel{1}\,e^1+1} \frac{u-1}{u}\,du$$

$$\frac{u}{u} - \frac{1}{u} = \boxed{1 - \frac{1}{u}}$$

$$\Big[\,u - \ln u\,\Big]_2^{e^1+1} = \left(e^1+1-\ln(e^1+1)\right) - \left(2 - \ln 2\right)$$

$$= e^1 - 1 - \ln\left(\frac{e^1+1}{2}\right)$$

$\boxed{5}$

7

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

(i) $x = 1 + 3\lambda \qquad y = -2 + \lambda$

$x = -8 + \mu \qquad y = 2 - 2\mu$

$1 + 3\lambda = -8 + \mu \qquad -2 + \lambda = 2 - 2\mu$

$3\lambda - \mu = -9 \qquad \lambda + 2\mu = 4$

$3\lambda - \mu = -9 \quad (\times 2)$

$\lambda + 2\mu = 4$

$+ \quad 6\lambda - 2\mu = -18$

$7\lambda = -14$

$\underline{\underline{\lambda = -2}}$

$3 \times -2 - \mu = -9$

$-6 - \mu = -9$

$\underline{\underline{\mu = 3}}$

$z = 4 + a\lambda$

$z = 3 - \mu$

$4 + a\lambda = 3 - \mu$

$4 - 2a = 3 - 3$

$4 - 2a = 0$

$\underline{\underline{a \neq 2}}$

$\boxed{6}$

(ii) $a = 2 \quad \lambda = -2 \quad \mu = 3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{matrix} 1 + -2 \times 3 \\ -2 + -2 \times 1 \\ 4 + -2 \times 2 \end{matrix} = \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix}$$

$\boxed{2}$

8 (i) $\int \cos^2 6x \, dx = \frac{1}{2}x + \frac{1}{24}\sin 12x + c$

double angle

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = \cos^2 A - (1 - \cos^2 A) \quad (\sin^2 A = 1 - \cos^2 A)$

$\cos 2A = \cos^2 A - 1 + \cos^2 A$

$\cos 2A = 2\cos^2 A - 1$

$\cos^2 A = \frac{1}{2}\cos 2A + \frac{1}{2}$

$\int \frac{1}{2}\cos 2(6x) + \frac{1}{2} \, dx$

$\int \frac{1}{2}\cos 12x + \frac{1}{2} \, dx$

$\left[\frac{1}{24}\sin 12x + \frac{1}{2}x + c\right]$ $\boxed{3}$

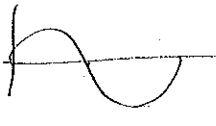
(ii) parts $\frac{du}{dx} = 1$ $\quad u = x \quad v = \frac{1}{24}\sin 12x + \frac{1}{2}x \quad \frac{dv}{dx} = \cos^2 6x$

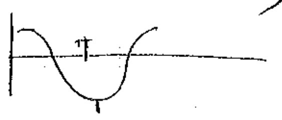
$\int x\cos^2 6x \, dx = \cancel{\frac{1}{24}\sin 12x + \frac{1}{2}x - \int}$

$= \left[x\left(\frac{1}{24}\sin 12x + \frac{1}{2}x\right)\right]_0^{\frac{1}{12}\pi} - \int \frac{1}{24}\sin 12x + \frac{1}{2}x \, dx$

$= \frac{\pi}{12}\left(\frac{\pi}{24}\right) - \left[-\frac{1}{288}\cos 12x + \frac{1}{4}x^2\right]_0^{\frac{1}{12}\pi}$ $\boxed{6}$

$= \frac{\pi^2}{288} - \left(\left(\frac{1}{288} + \frac{\pi^2}{576}\right) - \left(-\frac{1}{288}\right)\right) = \frac{\pi^2}{288} - \frac{1}{288} - \frac{\pi^2}{576} - \frac{1}{288}$





$\boxed{\frac{\pi^2}{576} - \frac{1}{144}}$ $= \frac{\pi^2}{288} - \frac{\pi^2}{576} - \frac{1}{144}$

9 (i) $x = 4\cos t$ $\quad\quad$ $y = 3\sin t$

$$\frac{dx}{dx} = -4\sin t \quad\quad \frac{dy}{dt} = 3\cos t$$

$$\frac{dy}{dx} = -\frac{3\cos t}{4\sin t}$$

$\boxed{3}$

(ii)

$$y = \frac{-3\cos p}{4\sin p}x + c$$

$$3\sin p = -\frac{3\cos p}{4\sin p}(4\cos p) + c$$

$$3\sin p = \frac{-12\cos^2 p}{4\sin p} + c$$

$$3\sin p + \frac{12\cos^2 p}{4\sin p} = c$$

$$y = -\frac{3\cos p}{4\sin p}x + 3\sin p + \frac{12\cos^2 p}{4\sin p}$$ $\times 4\sin p$

$$4y\sin p = -3\cos p\, x + 12\sin^2 p + 12\cos^2 p$$

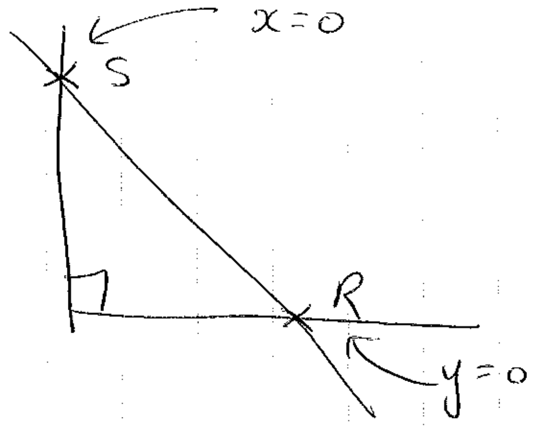
$$3x\cos p\, + 4y\sin p = 12\sin^2 p + 12\cos^2 p$$

$$3x\cos p + 4y\sin p = 12\cancel{(\sin^2 p + \cos^2 p)}$$

$\times 1$

3

(iii)



when $y=0$ $\qquad 3x\cos p = 12$

$$x = \frac{12}{3\cos p}$$

when $x=0$ $\qquad 4y\sin p = 12$

$$y = \frac{12}{4\sin p}$$

$$\triangle = \frac{1}{2}\left(\frac{12}{3\cos p} \times \frac{12}{4\sin p}\right)$$

$$= \frac{144}{24\cos p\sin p} = \frac{6}{\cos p \sin p}$$

$\sin 2A = 2\cos A\sin A$

$\frac{1}{2}\sin 2A = \cos A\sin A$

$$= \frac{6}{\frac{1}{2}\sin 2p}$$

$$= \frac{12}{\sin 2p}$$

$\boxed{3}$

(iv) $\frac{12}{\sin 2p} = A$

$0 < \sin 2p \leq 1$

least are $= 12$

when $p = \frac{1}{4}\pi$ rad

OR 45°

$\sin 2p = 0$

$2p = \sin^{-1} 0$

$2p = 0$

$p = 0$

$p \neq 0$

$\sin 2p = 1$

~~SIN~~

$2p = \sin^{-1}(1)$

$2p = \pi/2$

$p = \pi/4$

$\boxed{3}$