Further Maths Core Pure Maths 2

October 2020

① $y = 31\sinh x - 2\sinh 2x$ $\quad x \in \mathbb{R}$

$\frac{dy}{dx} = 31\cosh x - 4\cosh 2x$

$0 = 31\cosh x - 4\cosh 2x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\sinh^2 x = \cosh^2 x - 1$

So $\cosh 2x = 2\cosh^2 x - 1$

So $0 = 31\cosh x - 8\cosh^2 x + 4$

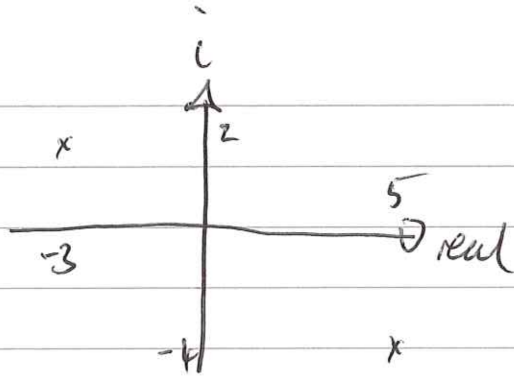
$\cosh x = 4$ or $-1/8$

$x = \ln(4+\sqrt{15})$ or $\ln(-1/8 + \sqrt{-63/64})$ not possible

$x = \ln(4+\sqrt{15})$

also can have $x = -\ln(4+\sqrt{15})$

(2)



centre of circle $\frac{-3+5}{2} = 1$, $\frac{2+-4}{2} = -1$ $\Big\}$ $1 - i$

length of diameter $\sqrt{(5--3)^2 + (-4-2)^2}$
$= 10$

radius $= 10/2 = 5$

so $|z - 1 + i| = 5$

③

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$$

ⓐ auxiliary function

$$100m^2 + 60m + 13 = 0$$

$$m = \frac{-3 \pm 2i}{10}$$

let $x = e^{\left(\frac{-3-2i}{10}\right)t} + e^{\left(\frac{-3+2i}{10}\right)t}$

$$= e^{-0.3t}\left(e^{-0.2it} + e^{+0.2it}\right)$$

$$= e^{-0.3t}\left(A\cos 0.2t + B\sin 0.2t\right)$$

let $13\lambda = 26 \Rightarrow \lambda = 2$

So ~~$x=$~~ $x = e^{-0.3t}\left(A\cos 0.2t + B\sin 0.2t\right) + 2$

$t = 0 \quad x = 0 \quad \frac{dx}{dt} = 10$

$0 = A + 2 \Rightarrow A = -2$

3 (b) $$\frac{dx}{dt} = -0.3e^{-0.3t}(A\cos 0.2t + B\sin 0.2t)$$

$$+ e^{-0.3t}(-0.2A\sin 0.2t + 0.2B\cos 0.2t)$$

So $$10 = -0.3(-2) + (0.2B)$$

$$9.4 = 0.2B \Rightarrow \underline{\underline{B = 47}}$$

So $$\frac{dx}{dt} = e^{-0.3t}\left(10\cos 0.2t - \frac{137}{10}\sin 0.2t\right)$$

Max x $$0 = e^{-0.3t}\left(10\cos 0.2t - \frac{137}{10}\sin 0.2t\right)$$

never zero

$$0 = 10\cos 0.2t - \frac{137}{10}\sin 0.2t$$

$$\frac{\sin 0.2t}{\cos 0.2t} = \frac{10 \times 10}{137} \Rightarrow t = \frac{\tan^{-1}(100/137)}{0.2}$$

$$t = 3.152650691$$

$$t = 3.15 \text{ weeks.}$$

Sub into x formulae $$x = e^{-0.3t}(-2\cos 0.2t + 47\sin 0.2t) + 2$$

$$x = 12.13433342$$

3c Second dose < 5 $\mu g/ml$

sub in again

$t=10$ $x= \ldots = 4.16918636$

As less than 5 $\mu g/ml$ safe to give second dose.

(4a) de Moivres $(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$

$$(\cos\theta + i\sin\theta)^7 = \cos^7\theta + \binom{7}{1}\cos^6\theta(i\sin\theta) + \dots$$

$$\text{so} = C^7 + 7C^6 iS - 21C^5S^2 - 35C^4iS^3$$
$$+35C^3S^4 + 21C^2iS^5 - 7CS^6 - iS^7$$

just compare imaginary coefficients

$$\sin 7\theta = 7C^6S - 35C^4S^3 + 21C^2S^5 - S^7$$
$$= 7(1-S^2)^3S - 35(1-S^2)^2S^3 + 21(1-S^2)S^5 - S^7$$
$$= 7(1-3S^2+3S^4-S^6)S - 35(1-2S^2+S^4)S^3 + 21(1-S^2)S^5 - S^7$$

$$\text{so } \sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

(b) let $x = \sin\theta$ then $1 + \sin 7\theta = 0$

$$\sin 7\theta = -1$$

$$7\theta = 3\pi/2,\ 7\pi/2,\ \dots$$
$$\theta = 3\pi/14,\ 7\pi/14,\ 11\pi/14,\ 15\pi/14,\ 19\pi/14,$$
$$= 0.623,\ 1\ \ 0.623\ \ -0.223,\ -0.901$$

5a) $y = \tan^{-1} x$

$\tan y = x$ differentiate w r t x

$\sec^2 y = \frac{dx}{dy}$

$\tan^2 y + 1 = \frac{dx}{dy}$

$x^2 + 1 = \frac{dx}{dy}$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

b) $\int f(x)\,dx = \int x \tan^{-1}(4x)\,dx$

integrate differentiate

integration by parts $\int u \frac{dv}{dx} dx = [uv] - \int v \frac{du}{dx} dx$

let $u = \tan^{-1}(4x)$ $v = \frac{x^2}{2}$

$\frac{du}{dx} = \frac{4}{1+(4x)^2}$ $\frac{dv}{dx} = x$

$\frac{x^2}{2} \tan^{-1}(4x) - \int \frac{2x^2}{1+16x^2} dx$

5b $\frac{x^2}{2}\tan^{-1}(4x) - \int \frac{1/8\,(16x^2)}{1+16x^2}\,dx$

$\frac{x^2}{2}\tan^{-1}(4x) - 1/8\int \frac{16x^2+1 \quad -1}{1+16x^2}\,dx$

$\frac{x^2}{2}\tan^{-1}(4x) - 1/8\int 1 \quad - \frac{1}{1+16x^2}\,dx$

$\frac{x^2}{2}\tan^{-1}(4\cancel{x}) - 1/8\int 1\,dx + 1/8\int 1/\cancel{1}16\,(1/16+x^2)\,dx$

$\frac{x^2}{2}\tan^{-1}(4x) - \frac{x}{8} + \frac{1}{128}\int \frac{1}{1/16+x^2}\,dx$

$\frac{x^2}{2}\tan^{-1}(4x) - \frac{x}{8} + \frac{1}{128}\left(\cancel{1}4\,\tan^{-1}4x\right) + k$

$\frac{x^2}{2}\tan^{-1}(4x) - \frac{x}{8} + \frac{1}{32}\tan^{-1}4x \quad + k$

(c) Mean Value $= \frac{1}{b-a}\int x f(x)\,dx$

$\left[\frac{1}{\frac{\sqrt{3}}{4}-0}\right]\left[\frac{x^2}{2}\tan^{-1}4x - x/8 + 1/32\tan^{-1}4x\right]_0^{\sqrt{3}/4}$

$= \frac{\sqrt{3}\,\pi}{18} - 1/8$

6 a

$$M = \begin{pmatrix} k & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$|M| = k(-1-1) - 5(-1-2) + 7(1-2)$$
$$= -2k + 8$$

Minors $\begin{pmatrix} -2 & -3 & -1 \\ -12 & -k-14 & k-10 \\ -2 & k-7 & k-5 \end{pmatrix}$

$$\begin{pmatrix} -2 & 3 & -1 \\ 12 & -k-14 & 10-k \\ -2 & 7-k & k-5 \end{pmatrix}$$

$$M^{-1} \frac{1}{-2k+8} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -k-14 & 7-k \\ -1 & 10-k & k-5 \end{pmatrix}$$

6b

$$M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

$$M^{-1} M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-4+8} \begin{pmatrix} -2 & 12 & -2 \\ 3 & -16 & 5 \\ -1 & 8 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$$

$$x = \frac{-6+12p}{4} = -\frac{3}{2} + 3p$$

$$y = \frac{13-16p}{4}$$

$$z = \frac{-7 + \overset{8}{\cancel{12}}p}{4}$$

$$\left(-\frac{3}{2} + 3p \,,\; \frac{13}{4} - 4p \,,\; -\frac{7}{4} + 2p \right)$$

6 c (i) Intersect value of q

$$\begin{array}{r} 4x + 5y + 7z = 1 \\ - \quad 4x + 4y + 4z = 4q \\ \hline y + 3z = 1 - 4q \end{array} \qquad \begin{array}{r} 2x + y - z = 2 \\ - \quad 2x + 2y + 2z = 2q \\ \hline -y - 3z = 2 - 2q \\ y + 3z = 2q - 2 \end{array}$$

So $1 - 4q = 2q - 2 \Rightarrow q = 1/2$

(ii)
$$4x + 5y + 7z = 1$$
$$x + y + z = 1/2$$
$$2x + y - z = 2$$

let $x = 0$
$$\begin{array}{r} y + z = 1/2 \\ + \quad y - z = 2 \\ \hline 2y = 5/2 \\ y = 5/4 \end{array}$$
$(0, 5/4, -3/4)$

let $y = 0$
$$\begin{array}{r} x + z = 1/2 \\ + \quad 2x - z = 2 \\ \hline 3x = 5/2 \\ x = 5/6 \\ z = -1/3 \end{array}$$
$(5/6, 0, -1/3)$

$$\begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix} - \begin{pmatrix} 5/6 \\ 0 \\ -1/3 \end{pmatrix} = \begin{pmatrix} -5/6 \\ 5/4 \\ -5/12 \end{pmatrix} \Rightarrow \begin{pmatrix} -10 \\ 15 \\ -5 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

So $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 5/4 \\ -3/4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

(7) (a) C_1 $x = \dfrac{a}{y+b}$ $0.5 \leq y \leq 2.5$

y axis x axis $x=1$ C_1 and C_2

at A $\overset{x}{(1}, \overset{y}{0.5)}$ $1 = \dfrac{a}{0.5+b} \Rightarrow a = 0.5+b$

B $(0.5, 2.5)$ $0.5 = \dfrac{a}{2.5+b} \Rightarrow a = 1.25 + 0.5b$

$$\begin{array}{r} a = 1.25 + 0.5b \\ - \quad a = 0.5 + b \\ \hline 0 = 0.75 - 0.5b \end{array} \Rightarrow b = 1.5 \Rightarrow a = 2$$

(b) C_2 arc centre $(0, 3)$

$$(x-0)^2 + (y-3)^2 = r^2$$

$$(0.5)^2 + (-0.5)^2 = r^2 \Rightarrow r = 1/\sqrt{2}$$

So $x^2 + (y-3)^2 = 0.5$

Note when $x=0$ $(y-3)^2 = 1/2$

$y = 3 + 1/\sqrt{2}$

7b Volume $\pi \int x^2 \, dy$

$$\pi \left[\int \left(1 + \left(\frac{2}{y+1.5}\right)^2 + 0.5 - (y-3)^2\right) dy \right.$$

$$\pi \int_0^{1/2} 1 \, dy + \pi \int_{1/2}^{2\frac{1}{2}} \left(\frac{2}{(y+1.5)}\right)^2 dy$$

$$\pi \int_{2\frac{1}{2}}^{3+1/\sqrt{2}} \left(1/2 - (y-3)^2\right) dy$$

$$\pi\left([y]_0^{1/2} + 4\int_{1/2}^{2\frac{1}{2}} (y+1.5)^{-2} dy + \int_{2\frac{1}{2}}^{3+1/\sqrt{2}} -y^2 + 6y - 8.5 \, dy \right)$$

$$\pi\left([y]_0^{1/2} + 4\left[-(y+1.5)^{-1}\right]_{1/2}^{2\frac{1}{2}} + \left[-\frac{y^3}{3} + 3y^2 - 8.5y\right]_{2\frac{1}{2}}^{3+1/\sqrt{2}} \right)$$

$$= \tfrac{1}{2}\pi + \pi + \pi\left(5/24 + \sqrt{2}/6\right)$$

$$= \pi\left(41/24 + \sqrt{2}/6\right) \approx 6.11 \text{ cm}^3$$