Core 4 - June 2009 Miss Watson's solutions

(1)

$$
\begin{array}{r}
3x^2 - 4x - 5 \\
x^2 + x + 2 \;\overline{)\; 3x^4 - x^3 - 3x^2 - 14x - 8} \\
\underline{3x^4 + 3x^3 + 6x^2} \\
0 \;\; -4x^3 - 9x^2 - 14x - 8 \\
\underline{-4x^3 - 4x^2 - 8x} \\
0 \;\; -5x^2 - 6x - 8 \\
\underline{-5x^2 - 5x - 10} \\
0 \;\; -x + 2
\end{array}
$$

Quotient $= 3x^2 - 4x - 5$

Remainder $= -x + 2$

$\boxed{4}$

(2)

$$\int_1^{\sqrt{3}} \frac{1 - \tan^2\theta}{1 + \tan^2\theta} \, dx$$

use identity
$\tan^2\theta + 1 = \sec^2\theta$
$\tan^2\theta = \sec^2\theta - 1$

$x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta \, d\theta$

change limits

$x = \tan\theta$

$\theta = \tan^{-1}(x)$

$\theta = \tan^{-1}(\sqrt{3}) = \pi/3$

$\theta = \tan^{-1}(1) = \pi/4$

$$\int_{\pi/4}^{\pi/3} \frac{1 - (\sec^2\theta - 1)}{1 + (\sec^2\theta - 1)} \times \sec^2\theta \, d\theta$$

$$\int_{\pi/4}^{\pi/3} \frac{2 - \sec^2\theta}{\cancel{\sec^2\theta}} \times \cancel{\sec^2\theta} \, d\theta$$

$$\int_{\pi/4}^{\pi/3} 2 - \sec^2\theta \, d\theta = \Big[2\theta - \tan\theta \Big]_{\pi/4}^{\pi/3}$$

$$= \left(\frac{2\pi}{3} - \sqrt{3}\right) - \left(\frac{2\pi}{4} - 1\right) = \frac{\pi}{6} - \sqrt{3} + 1$$

$\boxed{7}$

3 (i) $(a+x)^{-2} = \left(a\left(1+\frac{x}{a}\right)\right)^{-2} = \frac{1}{a^2}\left(1+\frac{x}{a}\right)^{-2}$

← need in the for $(1+x)^n$
formula page 2, replace x with $\frac{x}{a}$ and $n=-2$

$1 + (-2)\left(\frac{x}{a}\right) + \frac{-2 \times -3}{1.2}\left(\frac{x}{a}\right)^2$

$1 - \frac{2x}{a} + \frac{3x^2}{a^2} = \left(1+\frac{x}{a}\right)^{-2}$

$\times \frac{1}{a^2}$

$\underline{\underline{\frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4}}} = \frac{1}{a^2}\left(1+\frac{x}{a}\right)^{-2}$ $\boxed{4}$

(ii)

	$\frac{1}{a^2}$	$-\frac{2x}{a^3}$	$+\frac{3x^2}{a^4}$
1	×	×	$\frac{3x^2}{a^4}$
$-x$	×	$+\frac{2x^2}{a^3}$	×

$\frac{3x^2}{a^4} + \frac{2x^2}{a^3} = 0x^2$

$\frac{3}{a^4} + \frac{2}{a^3} = 0$

$\frac{3}{a^4} + \frac{2a}{a^4} = 0$

$\frac{3+2a}{a^4} = 0$

$3+2a = 0$

$2a = -3$

$\underline{\underline{a = -3/2}}$ $\boxed{3}$

4 (i) $e^x(\sin 2x - 2\cos 2x)$ "Product rule"

$e^x(2\cos 2x + 4\sin 2x) + e^x(\sin 2x - 2\cos 2x)$

$e^x(5\sin 2x)$

$\underline{\underline{5e^x \sin 2x}}$ $\boxed{4}$

4 (ii) $\int_0^{\frac{1}{4}\pi} e^x \sin 2x \, dx = \frac{1}{5}\int_0^{\frac{1}{4}\pi} 5e^x \sin 2x \, dx$

$\frac{1}{5}\left[e^x(\sin 2x - 2\cos 2x) \right]_0^{\frac{1}{4}\pi}$

$\frac{1}{5}\left\{ \left(e^{\frac{1}{4}\pi}\left(\sin\left(\tfrac{1}{2}\pi\right) - 2\cos\left(\tfrac{1}{2}\pi\right)\right)\right) - \left(e^0(\sin(0) - 2\cos(0))\right) \right\}$

$\frac{1}{5}\left\{ \left(e^{\pi/4}(1-0)\right) - \left(1 \times (0-2)\right) \right\}$

$\underline{\underline{\frac{1}{5}e^{\pi/4} + \frac{2}{5}}}$ $\boxed{3}$

5 (i) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t + 3t^2}{2 + 2t}$ at $(3, -9)$

$3 = 2t + t^2 \leftarrow 0 = t^2 + 2t - 3$

$-9 = 2t^2 + t^3 \quad 0 = (t-1)(t+3)$

$t = 1$ or $t = -3$

$\times \qquad \checkmark$

$\frac{dy}{dx} = \frac{4(-3) + 3(-3)^2}{2 + 2(-3)} = \underline{\underline{\frac{15}{-4}}}$ $\boxed{5}$

(ii) $\frac{y}{x} = \frac{2t^2 + t^3}{2t + t^2} = \frac{2t + t^2}{2 + t} = \frac{t\cancel{(2+t)}}{\cancel{(2+t)}} = t$

$\frac{y}{x} = t$

$y = 2\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3$

$y = \frac{2y^2}{x^2} + \frac{y^3}{x^3} \quad (\times x^3)$

$x^3 y = 2xy^2 + y^3 \quad (\div y)$ $\boxed{4}$

$\underline{\underline{x^3 = 2xy + y^2}}$

6 (i) $\dfrac{4x}{(x-5)(x-3)^2} = \dfrac{A}{x-5} + \dfrac{B}{x-3} + \dfrac{C}{(x-3)^2}$

* sort A out $\times(x-5)$ set $x=5$

$\dfrac{4x}{(x-3)^2} = A + 0 + 0$

$A = 5$

* sort C out $\times(x-3)^2$ set $x=3$

$\dfrac{4x}{(x-5)} = 0 + 0 + C$

$c = -6$

* sort B out, set $x=0$ and $A=5$ $C=-6$

$0 = \dfrac{5}{-5} + \dfrac{B}{-3} + \dfrac{-6}{9}$

$B = -5$

$\dfrac{4x}{(x-5)(x-3)^2} = \dfrac{5}{x-5} - \dfrac{5}{x-3} - \dfrac{6}{(x-3)^2}$ $\boxed{4}$

(ii) $\displaystyle\int_1^2 \frac{5}{x-5} - \frac{5}{x-3} - \frac{6}{(x-3)^2}\,dx$ $\quad 6(x-3)^{-2}$

$= \left[5\ln|x-5| - 5\ln|x-3| + \dfrac{6}{x-3}\right]_1^2$

$= \left(5\ln|-3| - 5\ln|-1| + \frac{6}{-1}\right) - \left(5\ln|-4| - 5\ln|-2| + \frac{6}{-2}\right)$

$= (5\ln 3 - 5\ln 1 - 6) - (5\ln 4 - 5\ln 2 - 3)$

$= (5\ln 3 - 6) - (5\ln\frac{4}{2} - 3)$

$= 5\ln\frac{3}{2} - 3$ $\boxed{5}$

7 (i) $u = \begin{pmatrix} 3/13 \\ b \\ c \end{pmatrix}$ $r = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

$u.r = 0 = 12/13 + 0 + c$ $n.w = 0 = \frac{12}{13} + 3b + {-24}/13$

$\underline{\underline{c = -12/13}}$ $\underline{\underline{b = \frac{4}{13}}}$

u is a unit vector

$|n| = \sqrt{(3/13)^2 + (\frac{4}{13})^2 + (-12/13)^2} = 1$ $\boxed{6}$

(ii) $r = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

$|r| = \sqrt{4^2+0^2+1^2}$ $= \sqrt{17}$ $|w| = \sqrt{4^2+3^2+2^2}$ $= \sqrt{29}$ $r.w = 16 + 0 + 2$ $= 18$

$\cos\theta = \frac{18}{\sqrt{17} \times \sqrt{29}} = 0.8106742284$

$\theta = 35.83765295$

$\underline{\underline{\theta = 36^\circ}}$ $\boxed{3}$

product rule

8 (i) $14x^2 - 7xy + y^2 = 2$

$28x - 7x\frac{dy}{dx} + -7y + 2y\frac{dy}{dx} = 0$

$28x - 7y = 7x\frac{dy}{dx} - 2y\frac{dy}{dx}$

$28x - 7y = \frac{dy}{dx}(7x - 2y)$

$\frac{28x - 7y}{7x - 2y} = \frac{dy}{dx}$ $\boxed{4}$

(ii) $x = 1 \Rightarrow 14 - 7y + y^2 = 2$

$y^2 - 7y + 12 = 0$

$(y-3)(y-4) = 0$ $(1, 3)$

$y = 3 \quad y = 4$ $(1, 4)$

continued...

(8)(ii)

$(1,3)$

$$\frac{dy}{dx} = \frac{28\times1-7\times3}{7\times1-2\times3} = 7$$

$(1,4)$

$$\frac{dy}{dx} = \frac{28\times1-7\times4}{7\times1-2\times4} = 0$$

$y = 7x + c \leftarrow (1,3)$

$3 = 7 + c$

$c = -4$

horizontal line at $y = 4$

$y = 7x - 4$

$4 = 7x - 4$

$x = 8/7$

coordinates of N

$\left(8/7, 4\right)$

$\boxed{6}$

(9)(i) $\frac{d\theta}{dt} = K_1 \leftarrow$ integrate with respect to t

$\theta = K_1t + c \leftarrow$ sub in initial conditions $\theta = 40$ $t = 0$

$c = 40$

$\theta = K_1t + 40 \leftarrow$ sub in $\theta = 60$

$60 = K_1t + 40$

$t = 20/K_1$

$\boxed{1}$

(ii) $\frac{d\theta}{dt} = -K_2(\theta - 20)$

$\boxed{1}$

need to flip this over so we can integrate w.r.t θ

(iii) $\frac{dt}{d\theta} = \frac{-1}{K(\theta-20)} \leftarrow$ take out constant $\frac{-1}{K}$ to make it easier to integrate.

$$t = \frac{-1}{K}\int \frac{1}{\theta - 20}\, d\theta$$

$t = -\frac{1}{K}\ln|\theta - 20| + c \leftarrow$ sub in initial conditions $t = 0$ $\theta = 60$ so you can find c.

$0 = -\frac{1}{K}\ln|40| + c$

$c = \frac{1}{K}\ln|40|$

$t = -\frac{1}{K}\ln|\theta - 20| + \frac{1}{K}\ln|40| \leftarrow$ sub in $\theta = 40$ and use log laws $\log a - \log b = \log\frac{a}{b}$

$t = -\frac{1}{K}\ln|20| + \frac{1}{K}\ln|40|$

$t = \frac{1}{K}\ln\left|\frac{40}{20}\right| = \frac{1}{K}\ln|2|$

Total time is $\frac{20}{K_1} + \frac{1}{K_2}\ln 2$

$\boxed{8}$