Pearson Edexcel

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Further Maths

Core Pure Maths 2

(i) Let $y = \tanh^{-1}(x)$ so $x = \tanh y \equiv \dfrac{e^{2y} - 1}{e^{2y} + 1}$

$\Rightarrow x(e^{2y} + 1) \equiv e^{2y} - 1$ so $xe^{2y} - e^{2y} = -1 - x$ $\Rightarrow e^{2y}(x-1) = -1 - x$

$\Rightarrow e^{2y} = \dfrac{-1 - x}{x - 1} = \dfrac{1 + x}{1 - x}$ $\Rightarrow 2y = \ln\left(\dfrac{1+x}{1-x}\right)$ $y = \frac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right) = \tanh^{-1}(x)$

for $-1 < x < 1$

(2). (i) $x^3 - 2x^2 + 4x - 5 = 0$ So $\frac{2}{p} + \frac{2}{q} + \frac{2}{r}$

$$2\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) = 2\left(\frac{qr + pr + pq}{pqr}\right) = 2\left(\frac{c/a}{-d/a}\right) = 2\left(\frac{c}{a} \times \frac{a}{-d}\right)$$

$$= \frac{2c}{-d} = \frac{2 \times 4}{5} = \frac{8}{5}$$

(ii) Each new root 4 less than the old root. So $w = x - 4 \Rightarrow x = w + 4$

$(w+4)^3 = w^3 + 3w^2(4) + 3(w^2)(4^2) + 4^3 = w^3 + 12w^2 + \cancel{48}w + 64$

$(w+4)^2 = w^2 + 8w + 16$

$$\begin{aligned} x^3 - 2x^2 + 4x - 5 = \quad & w^3 + 12w^2 + 48w + 64 \\ & \quad\;\; - 2w^2 - 16w - 32 \\ & \qquad\qquad + 4w + 16 \\ & \qquad\qquad\qquad - 5 \\ \hline & w^3 + 10w^2 + 36w + 43 \end{aligned}$$

$\alpha\beta\gamma = \frac{-d}{a} = \frac{-43}{1} = \underline{\underline{-43}}$

2(iii) $\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)+3\alpha\beta\gamma$

$\alpha+\beta+\gamma=\frac{-b}{a}=\frac{2}{1}=2$ $\quad$ $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=\frac{4}{1}=4$ $\quad$ $\alpha\beta\gamma=\frac{-d}{a}=\frac{5}{1}=5$

$\alpha^3+\beta^3+\gamma^3=2^3-3(2)(4)+3(5)=-1$

(3) Let $x = a\sinh u$ but $\cosh^2 u - \sinh^2 u \equiv 1$

so $a^2\cosh^2 u \equiv a^2 + a^2\sinh^2 u$

let $a = 3/2$

$$\int \frac{1}{\sqrt{4x^2+9}}\,dx \equiv \int \frac{1}{\sqrt{4(x^2+\frac{9}{4})}}\,dx \equiv \int \frac{1}{2\sqrt{x^2+9/4}}\,dx = 1/2\int \frac{1}{\sqrt{x^2+9/4}}\,dx$$

$$= 1/2 \int \frac{1}{a\cosh u}\,dx$$ but $x = a\sinh u$ $\frac{dx}{du} = -a\cosh u \Rightarrow dx = -a\cosh u\,du$

so $1/2\int \frac{1}{\cancel{a\cosh u}}(\cancel{-a\cosh u})\,du = 1/2\int du = 1/2\,u + K = 1/2\,\text{arsinh}\left(\frac{x}{a}\right) + K$

$$= 1/2\,\text{arsinh}\left(\frac{2x}{3}\right) + K$$

but $\text{arsinh}\, x = \ln(x + \sqrt{x^2+1})$

(b) $\left[1/2\,\text{arsinh}\left(\frac{2x}{3}\right)\right]_0^3$ $1/2(\ln(2 + \sqrt{5}) - \ln 1) = 1/2\ln(2+\sqrt{5})$

but mean value $\frac{1}{2} \times \frac{1}{3}\ln(2+\sqrt{5}) = \frac{1}{6}\ln(2+\sqrt{5})$

(4) (a) $C + iS = (\cos\theta + i\sin\theta) + (\tfrac{1}{2}\cos 5\theta + \tfrac{1}{2} i \sin 5\theta) + (\tfrac{1}{4}\cos 9\theta + \tfrac{1}{4} i \sin 9\theta) + \dots$

$= e^{i\theta} + \tfrac{1}{2} e^{5i\theta} + \tfrac{1}{4} e^{9i\theta} + \dots$

This is a decreasing GP, so the sum to infinity is $\dfrac{a}{1-r} = \dfrac{e^{i\theta}}{1 - \tfrac{1}{2}e^{4i\theta}} = \dfrac{2e^{i\theta}}{2 - e^{4i\theta}}$

(b) $\dfrac{2\cos\theta + 2i\sin\theta}{2 - (\cos 4\theta + i\sin 4\theta)} \equiv \dfrac{(2\cos\theta + 2i\sin\theta)}{(2 - \cos 4\theta - i\sin 4\theta)} \times \dfrac{(2 - \cos 4\theta + i\sin 4\theta)}{(2 - \cos 4\theta + i\sin 4\theta)}$ Just consider the co-efficient of i

$$\frac{2\cos\theta\sin 4\theta + 4\sin\theta - 2\sin\theta\cos 4\theta}{\cancel{4} - 2\cos 4\theta + 2i\cancel{\sin 4\theta} - 2\cos 4\theta + \cancel{\cos^2 4\theta} - i\cancel{\cos 4\theta}\sin 4\theta - 2i\cancel{\sin 2\theta} + i\cancel{\sin 4\theta}\cos 4\theta + \cancel{\sin^2} 4\theta}$$

$$\equiv \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(5) $\frac{4d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$ auxiliary equation $4m^2+4m+37=0$ No real roots

so $4(m^2+m)+37=0$ $4((m+\frac{1}{2})^2 - \frac{1}{4}) + 37 = 0$ $4(m+\frac{1}{2})^2 - 1 + 37 = 0$

$4(m+\frac{1}{2})^2 + 36 = 0$ $\Rightarrow (m+\frac{1}{2})^2 = -9$ so $m+\frac{1}{2} = \pm 3i$ so $m = -\frac{1}{2} \pm 3i$

Therefore general equation $y = e^{(-\frac{1}{2} \pm 3i)x}$ $y = e^{-\frac{1}{2}x}(A\cos 3x + B\sin 3x)$

$h = e^{(-\frac{1}{2} \pm 3i)t}$ $h = e^{-\frac{1}{2}t}(A\cos 3t + B\sin 3t)$

If $t=0$ $h=-20$ $\frac{dh}{dt} = 55$ from $h = e^{-\frac{1}{2}t}(A\cos 3t + B\sin 3t)$ $\Rightarrow A = -20$

$h = Ae^{-\frac{1}{2}t}\cos 3t + Be^{-\frac{1}{2}t}\sin 3t$

$\frac{dh}{dt} = -\frac{1}{2}Ae^{-\frac{1}{2}t}\cos 3t - 3Ae^{-\frac{1}{2}t}\sin 3t + -\frac{1}{2}Be^{-\frac{1}{2}t}\sin 3t + 3Be^{-\frac{1}{2}t}\cos 3t$

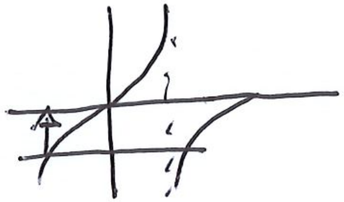
$t=0$ $\frac{dh}{dt} = -\frac{1}{2}A + 3B = 55$ so $10 + 3B = 55$ $3B = 45$ $\Rightarrow B = 15$

(5) $\frac{dh}{dt} = 0$ so $0 = -\tfrac{1}{2}Ae^{-\frac{1}{2}t}\cos 3t - 3Ae^{-\frac{1}{2}t}\sin 3t - \tfrac{1}{2}Be^{-\frac{1}{2}t}\sin 3t + 3Be^{-\frac{1}{2}t}\cos 3t$

$$\left(-\tfrac{1}{2}Ae^{-\frac{1}{2}t} + 3Be^{-\frac{1}{2}t}\right)\cos 3t = \left(\tfrac{1}{2}Be^{-\frac{1}{2}t} + 3Ae^{-\frac{1}{2}t}\right)\sin 3t$$

$$\frac{\sin 3t}{\cos 3t} = \frac{-\tfrac{1}{2}A\cancel{e^{-\frac{1}{2}t}} + 3B\cancel{e^{-\frac{1}{2}t}}}{\tfrac{1}{2}B\cancel{e^{-\frac{1}{2}t}} + 3A\cancel{e^{-\frac{1}{2}t}}} \Rightarrow \tan 3t = \frac{-22}{21}$$

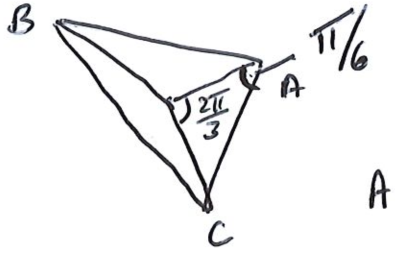
$3t = -0.808649786$ radians



so $3t = \pi - 0.8086497762$

$\underline{\underline{t = 0.778}}$

(6)(b) Area of ABC



so Area ABC =

$$A \to B = \sqrt{(6-(-3-\sqrt{3}))^2 + (2-(3\sqrt{3}-1))^2}$$

$$= \sqrt{84 + 18\sqrt{3} + 36 - 18\sqrt{3}}$$

$$= \sqrt{120} = 2\sqrt{30}$$

so area $\cancel{\tfrac{1}{2}} \times \cancel{2}\sqrt{30} \times 2\sqrt{30} \times \sin \pi/3 = 30\sqrt{3}$

But want midpoints

Area of triangle from midpoints is $1/4$ of the outered triangle

so $\dfrac{30\sqrt{3}}{4} = \dfrac{15\sqrt{3}}{2}$

(7a) Determinant $\begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix}$ $2\begin{vmatrix} k & 4 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & k \\ 3 & 2 \end{vmatrix}$

$2(-k-8)+(-3-12)+(6-3k) \neq 0$

$-2k-16-15+6-3k \neq 0 \quad \Rightarrow -25 \neq 5k \quad -5 \neq k$ so inverse provided that $k \neq -5$

(b) $M = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -6 & 4 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$ $\quad M^{-1} = \begin{pmatrix} -2/5 & 1/5 & 2/5 \\ 3 & -1 & -1 \\ 24/5 & -7/5 & -9/5 \end{pmatrix}$

so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2/5\,p + 1/5 \\ 3p - 1 \\ 24/5\,p - 7/5 \end{pmatrix}$ $\quad \underline{\underline{\left(-\tfrac{2}{5}p + \tfrac{1}{5},\ 3p-1,\ \tfrac{24}{5}p - \tfrac{7}{5}\right)}}$

c(i) $\begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ q \\ 0 \end{pmatrix}$ But from (a) no inverse possible, matrices don't meet at a point. Could have parallel planes, checked and not the case. could meet at a line

so $3x - 5y + 4z = q$

$\frac{6}{7} + \frac{15}{7} + 0 = q \Rightarrow q = 3$

Let $z = 0$ $\quad 2x - y = 1 \quad y = 2x - 1$

$3x + 2y = 0 \quad 3x + 2(2x-1) = 0$

$3x + 4x - 2 = 0$

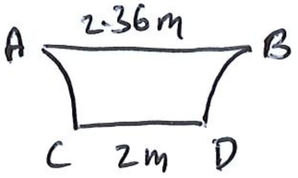
$7x = 2$

$x = 2/7,\ y = -3/7$

7(ii) When $q=3$ there are infinite solutions to these 3 equations as geometrically the planes meet at a line  so the solution set is the line.

(8)

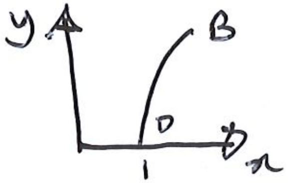
vertical



horizontal cross section

2.36 m diameter at the top 2m diameter at the bottom of the pool

$$y = \ln(3.6x - k) \qquad 1 \le x \le 1.18$$



(a) If $x = 1$ $y = 0$ so $0 = \ln(3.6x - k)$ but $\ln 1 = 0$ so $3.6 - k = 1 \Rightarrow k = 2.6$

(b) If $x = 1.18$ then $y = \ln(3.6 \times 1.18 - 2.6) = 0.4995624315$

(c) $$\frac{\pi}{3.6^2}\int_0^h \left(e^{2y} + 2ke^{y} + k^2\right)dy = \frac{\pi}{3.6^2}\left[\frac{e^{2y}}{2} + 2ke^{y} + yk^2\right]_0^h$$

$$= \frac{\pi}{3.6^2}\left(\frac{e^{2h}}{2} + 2ke^{h} + hk^2 - \left(\frac{e^{0}}{2} + 2ke^{0} + k^2 \times 0\right)\right) \text{ but } k = 2.6$$

$$= \frac{\pi}{3.6^2}\left(\frac{e^{2h}}{2} + \frac{26}{5}e^{h} + \frac{169}{25}h - \frac{1}{2} - \frac{26}{5}\right) = \frac{\pi}{3.6^2}\left(\frac{e^{2h}}{2} + \frac{26e^{h}}{5} + \frac{169h}{25} - 5.7\right)$$

(8)(d) $V = \frac{\pi}{3 \cdot 6^2}\left(\frac{e^{2h}}{2} + 2ke^{h} + hk^2\right) \Rightarrow \frac{dV}{dh} = \frac{\pi}{3 \cdot 6^2}\left(e^{2h} + 2ke^{h} + k^2\right)$

But $k = 2.6$ and depth of water is 0.2 m so $\frac{dV}{dh} = \frac{\pi}{3 \cdot 6^2}\left(e^{0.4} + 5.2e^{0.2} + 6.76\right)$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{3.539\ldots} \times 0.015 \times 60$ $\qquad \frac{dh}{dt} = 25.4 \text{ cm } h^{-1}$