(1) October 2020 Further Maths

Core Pure1

(1)(a) $f(z) = 3z^3 + pz^2 + 57z + q$

Roots are complex conjugates $3 - 2\sqrt{2}i$ α

$3 + 2\sqrt{2}i$ β

γ

Cubic rules $\alpha + \beta + \gamma = \frac{-b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ $\alpha\beta\gamma = \frac{-d}{a}$

$\frac{57}{3} = (3-2\sqrt{2}i)(3+2\sqrt{2}i) + (3-2\sqrt{2}i)\gamma + (3+2\sqrt{2}i)\gamma$

$19 = 9 + 8 + 6\gamma \Rightarrow \gamma = 1/3$

(b) $(z - (3-2\sqrt{2}i))(z-(3+2\sqrt{2}i))$

$$\left(\begin{array}{llll} z^2 - 3z & -2\sqrt{2}iz & & \\ -3z & & +9 & +6\sqrt{2}i \\ & +2\sqrt{2}iz & +8 & -6\sqrt{2}i \end{array}\right)$$

$z^2 - 6z + 17$ other root $\gamma = 1/3$ $(z - 1/3)$

use $3(z - 1/3) = 3z - 1$

$$(z^2 - 6z + 17)(3z - 1) = \begin{array}{llll} 3z^3 & -z^2 & & \\ & -18z^2 & +6z & \\ & & +51z & -17 \\ \hline 3z^3 & -19z^2 & +57z & -17 \end{array}$$

(2) 2a Because integrating over a discontinuity

(b) $\frac{1}{x(2x+5)} \equiv \frac{A}{x} + \frac{B}{2x+5}$

$1 = A(2x+5) + Bx$

if $x=0$ $A = 1/5$ $\quad x = -5/2$ $B = -2/5$

$$\int_1^{\infty} 1/5x - 2/5(2x+5) \quad dx$$

$$= \left[1/5 \ln|x| - 1/5 \ln|2x+5 \right]_1^{\infty}$$

$$= 1/5 \left[\ln\left(x/2x+5 \right) \right]_1^{\infty}$$

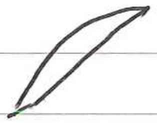
$= \cancel{1/5 \, (\ln(}$

$$= 1/5 \left(\lim_{x \to \infty} \left(\ln\left(x/2x+5 \right) \right) - \ln\left(1/2 \times 1+5 \right) \right)$$

$$= 1/5 \left(\ln(1/2) - \ln(1/7) \right)$$

$$= 1/5 \ln(7/2)$$

③ $\left(\begin{array}{l} \text{Area from intercept} \to \pi/2 \text{ under } C_1 \\ - \quad '' \quad '' \quad '' \quad '' \quad \text{under } C_2 \end{array} \right) \times 2$



Intercept $(1+\sin\theta) = 3(1-\sin\theta)$

$$1+\sin\theta = 3 - 3\sin\theta$$
$$4\sin\theta = 2$$
$$\sin\theta = 2/4 = 1/2$$
$$\theta = \sin^{-1}(1/2) = \pi/6$$

Area under curve $\frac{1}{2}r^2$

$$2\left(\frac{1}{2}\int_{\pi/3}^{\pi/2} (1+\sin\theta)^2 - (3(1-\sin\theta))^2 \, d\theta\right)$$

$$\int_{\pi/3}^{\pi/2} -8 + 20\sin\theta - 8\sin^2\theta \, d\theta$$

$$\cos(\theta+\theta) = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$2\sin^2\theta = 1 - \cos 2\theta$$
$$8\sin^2\theta = 4 - 4\cos 2\theta$$
$$-8\sin^2\theta = -4 + 4\cos 2\theta$$

$$\cancel{\tfrac{1}{2}} \int_{\pi/6}^{\pi/2} -8 + 20\sin\theta - 4 + 4\cos 2\theta \, d\theta$$

$$\int_{\pi/6}^{\pi/2} -12 + 20\sin\theta + 4\cos 2\theta \, d\theta$$

$$\left[-12\theta - 20\cos\theta + 2\sin 2\theta\right]_{\pi/6}^{\pi/2} = \left(-6\pi - 0 + 0 - (-2\pi - 10\sqrt{3} + \sqrt{3})\right)$$

$$= -4\pi + 9\sqrt{3}$$

4a $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

(1) $x = 2 + \lambda - \mu$
(2) $y = 4 + 2\lambda + 2\mu$
(3) $z = -1 - 3\lambda + \mu$

$2 \times (1) + (2)$ $\quad 2x + y = 8 + 4\lambda$ $\quad (4)$
$(2) - 2 \times (3)$ $\quad y - 2z = 6 + 8\lambda$ $\quad (5)$

$(5) - 2(4)$ $\quad -4x - y - 2z = -10$

or $\quad 4x + y + 2z = 10$

(b) $\dfrac{x-1}{5} = \dfrac{y-3}{-3} = \dfrac{z+2}{4} \Rightarrow \underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$

$4(1+5\lambda) + (3 - 3\lambda) + 2(-2 + 4\lambda) = 10$

So $\lambda = 7/25$ $\quad$ so $\left(12/5, 54/25, -22/25\right)$

(c) Normal to $4x + y + 2z = 10$ is $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

Angle between normals same as angle between the two planes $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 8 - 1 + 6 = 13$

$13 = \sqrt{4^2 + 1^2 + 2^2} \times \sqrt{2^2 + (-1)^2 + 3^2} \times \cos\theta \Rightarrow \underline{\underline{\theta = 41^\circ}}$

5(a) $\frac{dx}{dt} = -5x + 10y - 30 \quad \frac{d^2x}{dt^2} = -5\frac{dx}{dt} + 10\frac{dy}{dt}$

$$\frac{d^2x}{dt^2} = -5\frac{dx}{dt} + 10\frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = \begin{array}{l} -5(-5x+10y-30) \\ +10(-2x+3y-4) \\ +2(-5x+10y-30) \\ +5x \end{array} = \begin{array}{l} 25x - 50y + 150 \\ -20x + 30y - 40 \\ -10x + 20y - 60 \\ +5x \\ \hline \quad 50 \end{array}$$

(b) CF Auxiliary eqⁿ $m^2 + 2m + 5 = 0 \quad b^2 - 4ac < 0 \quad m = -1 \pm 2i$

So $x = Ae^{(-1+2i)t} + Be^{(-1-2i)t} = e^{-t}(Ae^{2it} + Be^{-2it}) =$
$= e^{-t}(P\cos 2t + Q\sin 2t)$

P.I. let $x = \lambda \quad \frac{dx}{dt} = 0 \quad \frac{d^2x}{dt^2} = 0 \quad 5\lambda = 50 \Rightarrow \lambda = 10$

So $x = e^{-t}(P\cos 2t + Q\sin 2t) + 10$

(c) $\frac{dx}{dt} = -e^{-t}(P\cos 2t + Q\sin 2t) + e^{-t}(-2P\sin 2t + 2Q\cos 2t)$
$= e^{-t}((-Q-2P)\sin 2t + ((2Q-P)\cos 2t)$

$\frac{dx}{dt} = -5x + 10y - 30 \quad y = \frac{1}{10}\left(\frac{dx}{dt} + 5x + 30\right)$

$y = \frac{1}{10}\left(e^{-t}\left(4Q - 2P\right)\sin 2t + \left(2Q + 4P\right)\cos 2t + 80\right)$

5d

$x = 2 \quad y=5 \quad t=0$

$2 = P+10 \Rightarrow P=-8$

$5 = \frac{1}{10}(2Q+4P+80)$

$50 = 2Q - 32 + 80 \Rightarrow Q=1$

so $x = e^{-t}(-8\cos 2t + \sin 2t) + 10$

$y = e^{-t}(2\sin 2t - 3\cos 2t) + 8$

(e) As e^{-8} is small 3.35×10^{-4}, then x and y remain around 10 and 8.

6(i) Base $k=1$

$$(3\times 1+1)(1+2) = 1(1+2)(1+3)$$
$$12 = 12 \quad \checkmark$$

Assume true for k^{th} term

Consider $k+1^{th}$ term

$$\sum_{r=1}^{k+1}(3r+1)(r+2) = k(k+2)(k+3) + (3(k+1)+1)((k+1)+2)$$

$$= k(k+2)(k+3) + (3k+4)(k+3)$$
$$\cancel{= (k+3)(k^2+2k+3k^2+9k+9k+12)}$$
$$\cancel{= (k+3)(4k^2+15k+12)}$$
$$\cancel{= (k+3)}$$
$$= (k+3)(k^2+2k+3k+4)$$
$$= (k+3)(k^2+5k+4)$$
$$= (k+3)(k+1)(k+4)$$
$$= (k+1)(k+3)(k+4)$$

If the statement is true for $n=k$ then it has been shown ~~to~~ true for $n=k+1$ and as it is true for $n=1$ the statement is true for all positive ingeters n.

(ii) when $n=1$ Base $4^1+5^1+6^1=15$

Assume true for $n=k$ $\quad 4^k+5^k+6^k$ is diverisible by 15

$$f(k+2)-f(k) = 4^{k+2}+5^{k+2}+6^{k+2}-(4^k+5^k+6^k)$$
$$= 15\ 4^k + 24\ 5^k + 35\ 6^k$$
$$= 15\ 4^k + 120\ 5^{k-1} + 210\ 6^{k-1}$$

So true for $n=1$ assumed true for $n=k$ shown difference between $n=k$ and $n=k+2$ also true, so true for all odd values.

(7) (a) $(1+t)\dfrac{dP}{dt} + P = t^{1/2}(1+t)$

Using inverse product rule

$$\frac{d}{dt}P(1+t) = t^{1/2}(1+t)$$

$$P(1+t) = \int t^{1/2} + t^{3/2}\,dt$$

$$P(1+t) = \frac{2}{3}t^{3/2} + \frac{2t^{5/2}}{5} + C$$

$$P = \frac{2/3\,t^{3/2} + 2/5\,t^{5/2} + C}{1+t}$$

$$P = \frac{10t^{3/2} + 6t^{5/2} + 75}{15(1+t)}$$

$t=8 \quad P = 10.2769643 \quad \Rightarrow \quad \underline{\underline{10,277}}$

(b) Sub $t=4$ into $P \quad \Rightarrow \quad P = 347/75$

Sub $t=4$ and $P = 347/75$ into $(1+t)\dfrac{dP}{dt} + P = t^{1/2}(1+t)$

gives $\dfrac{dP}{dt} = \dfrac{403}{375}$ need to $\times 1000$ gives 1075 bacteria per hour

(c) Bacteria will carry on increasing, this is unrealistic.