Core 4 June 2015.

i) $\dfrac{2}{3-x} + \dfrac{3}{1+x}$

$$= \frac{2(1+x) + 3(3-x)}{(3-x)(1+x)}$$

$$= \frac{2 + 2x + 9 - 3x}{(3-x)(1+x)}$$

$$= \frac{11 - x}{(3-x)(1+x)}$$

ii) $$\frac{\cancel{(11-x)}\cancel{(x+11)}(x-3)}{(3-x)(1+x)\cancel{(11-x)}\cancel{(x+11)}}$$

$$\frac{-\cancel{(3-x)}\ 1}{\cancel{(3-x)}(1+x)}$$

$$= \frac{-1}{1+x}$$

2) A $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ B $\begin{pmatrix} 5 \\ 9 \\ -5 \end{pmatrix}$ C $\begin{pmatrix} 6 \\ 5 \\ -4 \end{pmatrix}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ 8 \\ -8 \end{pmatrix}$

$\overrightarrow{AP} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}$ so $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\overrightarrow{OP} = \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix}$

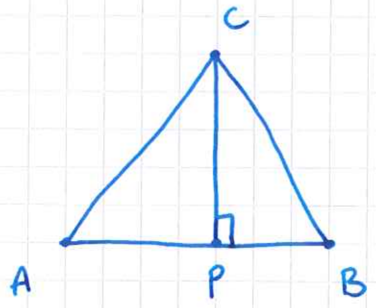
$\overrightarrow{CP} = -\underline{c} + \underline{p} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

$\underline{CP} \cdot \underline{AB} = \cancel{6\times 2+}\ 4 \times -2 + 8 \times 2 + -8 \times 1 = -8 + 16 - 8$

$= 0.$

dot product is 0 $\therefore$ perpendicular.

ii)



$$|AB| = \sqrt{16+64+64}$$
$$= 12$$
$$|CP| = \sqrt{4+4+1}$$
$$= 3.$$

so $\quad \text{area} = \frac{1}{2} \times 12 \times 3$
$$= \underline{18}.$$

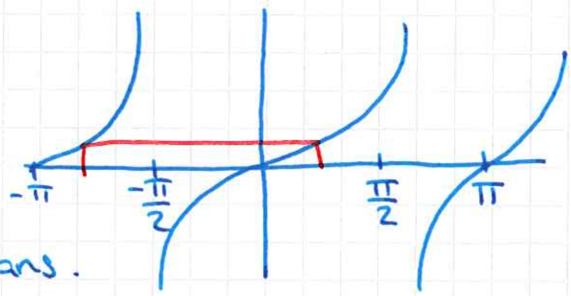
3i) $\quad y = e^{2x}\cos x$

$$\frac{dy}{dx} = 2e^{2x}\cos x - e^{2x}\sin x$$

$2e^{2x}\cos x - e^{2x}\sin x = 0 \quad$ at st pts.

$$e^{2x}(2\cos x - \sin x) = 0$$

e^{2x} is never 0



$$2\cos x - \sin x = 0$$
$$2\cos x = \sin x$$
$$\tan x = 2$$
$$x = 1.11 \text{ radians}.$$

or $\quad x = 1.11 - \pi = -2.03$ radians.

4i) $(8-9x)^{2/3} = 8^{2/3}\left(1-\frac{9x}{8}\right)^{2/3}$

$= 4\left(1 + \left(\frac{\cancel{2}}{\cancel{3}}\right)\left(-\frac{\overset{3}{\cancel{9}}x}{\underset{4}{\cancel{8}}}\right) + \frac{\left(\frac{\cancel{2}}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{9x}{8}\right)^2}{\cancel{2}}\right)$

$= 4\left(1 - \frac{3}{4}x - \frac{9x^2}{64}\right)$

$= 4 - 3x - \frac{9x^2}{16}$

ii) Valid for $\left|\frac{9x}{8}\right| < 1$

so $|x| < \frac{8}{9}$

5) $\int e^{2\sqrt{x+1}}\,dx$

$t = \sqrt{x+1} = (x+1)^{1/2}$

$\frac{dt}{dx} = \frac{1}{2}(x+1)^{-1/2}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{x+1}}$

$dx = 2\sqrt{x+1}\,dt$

$dx = 2t\,dt$

$= \int e^{2t} \times 2t\,dt$

$= \int \overset{v}{2t}\,\overset{u'}{e^{2t}}\,dt$

$v = 2t$ $v' = 2$

$u' = e^{2t}$ $u = \frac{e^{2t}}{2}$

$\int u'v = uv - \int uv'$

$= te^{2t} - \int 2e^{2t}$

$= te^{2t} - e^{2t} + C$

6i) $\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$ $\qquad\qquad$ $\frac{u'v - uv'}{v^2}$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

ii) $\int \frac{\sqrt{1+\cos 2x}}{\sin x \sin 2x}$

$$\sqrt{1+\cos 2x} = \sqrt{1 + (2\cos^2 x - 1)}$$

$$= \sqrt{2\cos^2 x}$$

$$= \sqrt{2}\cos x$$

$$\sin x \sin 2x = 2\sin^2 x \cos x$$

So $\int \frac{\sqrt{1+\cos 2x}}{\sin x \sin 2x} = \int \frac{\sqrt{2}\cancel{\cos x}}{2\sin^2 x \cancel{\cos x}}$

$$= \frac{\sqrt{2}}{2}\int \frac{1}{\sin^2 x}\, dx$$

$$= \left[-\frac{\sqrt{2}}{2}\frac{\cos x}{\sin x}\right]_{\pi/6}^{\pi/4}$$

$$= \left(-\frac{\sqrt{2}}{2}\frac{\cancel{\sqrt{2}/2}}{\cancel{\sqrt{2}/2}} - \left(-\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}/2}{1/2}\right)\right)$$

$$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}$$

$$= \frac{1}{2}\left(\sqrt{6} - \sqrt{2}\right)$$

7) $(x+y)^2 = xy^2$

$$x^2 + 2xy + y^2 = xy^2$$

$$2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(2x + 2y - 2xy\right) = y^2 - 2x - 2y.$$

$$\frac{dy}{dx} = \frac{y^2 - 2x - 2y}{2x + 2y - 2xy}.$$

when $x = 1$, find y

$$1 + 2y + y^2 = y^2$$

$$1 + 2y = 0$$

$$y = -\frac{1}{2}$$

so at $(1, -\frac{1}{2})$

$$\frac{dy}{dx} = \frac{\left(\frac{-1}{2}\right)^2 - 2 + 1}{2 - 1 + 1}$$

$$= \frac{\frac{1}{4} - 1}{2}$$

$$= \frac{-3}{8}$$

8) $\frac{dP}{dt} = 1$, when $t = 0$

$$\frac{dP}{dt} = \frac{k}{P}$$

$$\frac{dt}{dP} = \frac{P}{k}$$

$$\int k\,dt = \int P\,dP$$

$$kt = \frac{P^2}{2} + c$$

when $t = 0$, $P = 100$

$$0 = \frac{100^2}{2} + c$$

$$c = -5000$$

when $t = 0$, $\frac{dP}{dt} = 1$, $P = 100$

$$1 = \frac{k}{100}$$

$$k = 100$$

so $100t = \frac{P^2}{2} - 5000$

$$200t = P^2 - 10000$$

$$P^2 = 200t + 10000$$

$$P = \sqrt{200t + 10000}$$

when $t = 8$

$$P = 107.7 \approx 108$$

when $t = 13$

$$P = 112.2 .$$

The model is a good fit initial but, not as t increases as at $t = 13$

$$112.2 \not\approx 128 .$$

9) $\underline{r} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\quad \underline{r} = \begin{pmatrix} 4 \\ 10 \\ 19 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$

i) when the lines intersect at $(7,7,1)$

$$3 + 2\lambda = 7 \qquad 4 + \mu = 7$$
$$\lambda = 2 \qquad \mu = 3$$

when $\mu = 3$

$$19 + \mu a = 1$$
$$19 + 3a = 1$$
$$3a = -18$$
$$\underline{a = -6}.$$

ii) when $\theta = 60$ $\quad \underline{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$

$$\cos 60 = \frac{a \cdot b}{|a||b|}$$

$$\underline{a} \cdot \underline{b} = 2 - 1 + a = 1 + a$$

$$|\underline{a}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

$$|\underline{b}| = \sqrt{1 + 1 + a^2} = \sqrt{2 + a^2}$$

$$\frac{1 + a}{\sqrt{6}\sqrt{2 + a^2}} = \frac{1}{2}$$

$$2 + 2a = \sqrt{6}\sqrt{2 + a^2}$$
$$(2 + 2a)^2 = 6(2 + a^2)$$
$$4 + 8a + 4a^2 = 12 + 6a^2$$
$$2a^2 - 8a + 8 = 0$$
$$a^2 - 4a + 4 = 0$$
$$(a - 2)(a - 2) = 0$$
$$\underline{a = 2}.$$

10i) $\dfrac{x+8}{x(x+2)} = \dfrac{A}{x} + \dfrac{B}{x+2}$

$x+8 = A(x+2) + Bx$

let $x = 0$

$8 = 2A \longrightarrow \underline{A=4}.$

let $x = -2$

$6 = -2B \longrightarrow B = -3$

So $\dfrac{x+8}{x(x+2)} \quad \dfrac{4}{x} - \dfrac{3}{x+2}$

ii)

$$\begin{array}{r|l} & 7 \\ x^2+2x & 7x^2+16x+16 \\ & -\ \underline{7x^2+14x} \\ & \quad 2x+16 \end{array}$$

$7x^2+16x+16 = 7(x^2+2x) + 2x+16$

$= 7(x^2+2x) + 2(x+8)$

$\dfrac{7x^2+16x+16}{x^2+2x} = 7 + \dfrac{2(x+8)}{x^2+2x}$

$= 7 + \dfrac{8}{x} - \dfrac{6}{x+2}$

iii) $x = \dfrac{2t}{1-t}$ $\quad y = 3t + \dfrac{4}{t}$

$x - xt = 2t$

$x = 2t + xt$

$x = t(2+x)$

$t = \dfrac{x}{2+x}$

$y = \dfrac{3x}{2+x} + \dfrac{8+4x}{x}$

$$y = \frac{3x^2 + (8+4x)(2+x)}{x(x+2)}$$

$$= \frac{3x^2 + 16 + 16x + 4x^2}{x(x+2)}$$

$$= \cancel{3x^2+4}$$

$$\frac{7x^2 + 16x + 16}{x(x+2)}$$

iv) $$\int_1^2 \left(7 + \frac{8}{x} - \frac{6}{x+2}\right) dx$$

$$= \Big[7x + 8\ln x - 6\ln(x+2)\Big]_1^2$$

$$= (14 + 8\ln 2 - 6\ln 4) - (7 + \cancel{8\ln 1} - 6\ln 3)$$

$$= 14 + 8\ln 2 - 6\ln 4 - 7 + 6\ln 3$$

$$= 14 + 8\ln 2 - 6\ln 2^2 - 7 + 6\ln 3$$

$$= 7 + 8\ln 2 - 12\ln 2 + 6\ln 3$$

$$= \underline{7 - 4\ln 2 + 6\ln 3}.$$