Core 4 - Jan 2010 Miss Watson's solutions

(1)

$$\begin{array}{r|l} & x^2 + 6x - 4 \\ \hline x^2+5x+2 & x^4 + 11x^3 + 28x^2 + 3x + 1 \\ & \underline{x^4 + 5x^3 + 2x^2} \\ & 0 + 6x^3 + 26x^2 + 3x + 1 \\ & \underline{6x^3 + 30x^2 + 12x} \\ & 0 \quad - 4x^2 - 9x + 1 \\ & \underline{- 4x^2 - 20x - 8} \\ & 0 \quad 11x + 9 \end{array}$$

Quotient $= x^2 + 6x - 4$

Remainder $= 11x + 9$

$\boxed{4}$

(2) (i) $\overrightarrow{BA} = \begin{pmatrix} -5 \\ -10 \\ 12 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -12 \\ 15 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 3 \\ 6 \\ p \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ p+3 \end{pmatrix}$

$\begin{pmatrix} -6 \\ -12 \\ 15 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ p+3 \end{pmatrix} = 0$

$-12 - 48 + 15p + 45 = 0$

$15p - 15 = 0$

$\underline{p = 1}$

$\boxed{4}$

(ii) Parallel

$\begin{pmatrix} -6 \\ -12 \\ 15 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$ compare to $\begin{pmatrix} 2 \\ 4 \\ p+3 \end{pmatrix}$

$p + 3 = -5$

$\underline{p = -8}$

$\boxed{2}$

(3) Identity from double angle formulas

$\boxed{\cos 2A = 2\cos^2 A - 1}$

$$\frac{2\cos^2 x - 1}{\cos^2 x} = \frac{2\cos^2 x}{\cos^2 x} - \frac{1}{\cos^2 x} = 2 - \sec^2 x$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2 - \sec^2 x \, dx = \Big[2x - \tan x \Big]_{\pi/4}^{\pi/3}$$

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$$= \left(\frac{2\pi}{3} - \tan \pi/3\right) - \left(\frac{\pi}{2} - \tan \pi/4\right) = \left(\frac{2\pi}{3} - \sqrt{3}\right) - \left(\frac{\pi}{2} - 1\right)$$

$$= \frac{\pi}{6} - \sqrt{3} + 1$$ $\boxed{5}$

(4) $$\int_1^e \frac{1}{t \times u^2} \, dt$$

$u = 2 + \ln t$

$\frac{du}{dt} = \frac{1}{t}$

$t\,du = dt$

change limits

$u = 2 + \ln e = 3$

$u = 2 + \ln 1 = 2$

$$\int_2^3 \frac{1}{\cancel{t} \times u^2} \times \cancel{t}\, du = \int_2^3 u^{-2} \, du$$

$$= \Big[-u^{-1} \Big]_2^3 = \left(-\frac{1}{3}\right) - \left(-\frac{1}{2}\right) = \frac{1}{6}$$ $\boxed{6}$

(5) (i) $n = \frac{1}{3}$ green booklet page 2

$$(1+x)^{1/3} \approx 1 + \left(\tfrac{1}{3}\right)x + \frac{\frac{1}{3} \times -\frac{2}{3}}{1 \cdot 2}(x)^2$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2$$ $\boxed{2}$

(ii) $(8 + 16x)^{1/3} = (8(1+2x))^{1/3} = 2(1+2x)^{1/3}$

(i) $\times 2$ and replace x with $2x$

$$2 + \frac{2}{3}(2x) - \frac{2}{9}(2x)^2 = 2 + \frac{4}{3}x - \frac{8}{9}x^2$$ $\boxed{4}$

(5) (ii) (b) normally valid for $|x|<1$ but we replaced x with $2x$ so

$$|2x| < 1$$
$$\underline{\underline{|x| < 1/2}}$$

$\boxed{1}$

(6) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$y = t^3 - \ln(t^3)$ ← log laws $\ln a^b = b\ln a$ ← $3\ln t$

$\frac{dy}{dt} = 3t^2 - \frac{3}{t}$

$x = 9t - \ln(9t)$ ← log laws $\ln a \times b = \ln a + \ln b$ ← $-\ln t + \ln 9$ ↑ just a number so differentiates to nothing.

$\frac{dx}{dt} = 9 - \frac{1}{t}$

$\frac{dy}{dx} = \frac{3t^2 - \frac{3}{t}}{9 - \frac{1}{t}}$

$3 = \frac{3t^2 - \frac{3}{t}}{9 - \frac{1}{t}}$

$27 - \frac{3}{t} = 3t^2 - \frac{3}{t}$ $(\times t)$

$27t - 3 = 3t^3 - 3$

$0 = 3t^3 - 27t$

$0 = t(3t^2 - 27)$

$t = 0$ not valid as cannot take a log of zero.

$3t^2 - 27 = 0$

$t = \pm\sqrt{9}$

$\underline{\underline{t = 3}}$, $t = -3$ not valid as cannot take log of a negative.

$\boxed{6}$

(7) $x^3 + 2x^2y = y^3 + 15$ (differentiate) product rule

$3x^2 + 2x^2\frac{dy}{dx} + 4xy = 3y^2\frac{dy}{dx}$

$3x^2 + 4xy = 3y^2\frac{dy}{dx} - 2x^2\frac{dy}{dx}$

$3x^2 + 4xy = \frac{dy}{dx}(3y^2 - 2x^2)$

sub in $(2,1)$ → $\frac{3x^2 + 4xy}{3y^2 - 2x^2} = \frac{dy}{dx}$

$\frac{20}{-5} = \frac{dy}{dx} = -4$ ← tangent

normal gradient $= \frac{1}{4}$

continued

7 $y = \frac{1}{4}x + c \leftarrow (2,1)$

$1 = \frac{2}{4} + c \qquad c = \frac{1}{2}$

$y = \frac{1}{4}x + \frac{1}{2} \qquad (\times 4)$

$4y = x + 2$

$0 = x - 4y + 2$

$\boxed{8}$

8 (i) $e^{\cos x}$

differentiate $e^{\cos x} \times -\sin x = -\sin x\, e^{\cos x}$

$\boxed{1}$

(ii) Page 5 green booklet $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$

use part (i) to do this

$\frac{dv}{dx} = \sin x\, e^{\cos x} \qquad u = \cos x$

$v = -e^{\cos x} \qquad \frac{du}{dx} = -\sin x$

$\Rightarrow -\cos x\, e^{\cos x} - \int -e^{\cos x} \times -\sin x\, dx$

$\Rightarrow -\cos x\, e^{\cos x} - \int \sin x\, e^{\cos x}\, dx$

use part(i) again

$\Rightarrow \left[-\cos x\, e^{\cos x}\right]_0^{\pi/2} - \left[-e^{\cos x}\right]_0^{\pi/2}$

$\left(-\cos\left(\frac{\pi}{2}\right)e^{\cos \pi/2}\right) - \left(-\cos 0\, e^{\cos 0}\right)$ | $\left(-e^{\cos \pi/2}\right) - \left(-e^{\cos 0}\right)$

$0 + e$ | $-1 + e$

$e - (-1 + e) = 1$

$\boxed{6}$

OR $\left[-\cos x\, e^{\cos x} + e^{\cos x}\right]_0^{\pi/2}$

$= (0 + 1) - (-e + e) = 1$

9 (i) $P = \begin{pmatrix} 3+1 \\ 1-1 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ direction vector $= d = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$|P| = \sqrt{4^2+0^2+3^2} = 5$ $|d| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$

$P \cdot d = 4 + 0 + 6 = 10$

$\cos\theta = \dfrac{10}{5\sqrt{6}}$

$\theta = 0.615$ RAD

OR $\theta = 35.3°$

$\boxed{4}$

(ii) $Q = \begin{pmatrix} 3+t \\ 1-t \\ 1+2t \end{pmatrix}$ $d = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$Q \cdot d = 0$

$1(3+t) + -1(1-t) + 2(1+2t) = 0$

$3 + t - 1 + t + 2 + 4t = 0$

$6t = -4$

$t = -2/3$

$Q = \begin{pmatrix} 3 - 2/3 \\ 1 - -2/3 \\ 1 + 2 \times \frac{-2}{3} \end{pmatrix} = \begin{pmatrix} 7/3 \\ 5/3 \\ -1/3 \end{pmatrix}$

$\boxed{4}$

(iii) $|Q| = \sqrt{(7/3)^2 + (5/3)^2 + (-1/3)^2} = \dfrac{5\sqrt{3}}{3}$

$\boxed{2}$

10 (i) $\dfrac{1}{(3-x)(6-x)} = \dfrac{A}{(3-x)} + \dfrac{B}{(6-x)} = \dfrac{A(6-x) + B(3-x)}{(3-x)(6-x)}$

$= \dfrac{6A - xA + 3B - xB}{(3-x)(6-x)}$

① $6A + 3B = 1$

$-A - B = 0$ $\times 3$

② $-3A - 3B = 0$

$= \dfrac{1/3}{(3-x)} - \dfrac{1/3}{(6-x)}$ ① + ② $\Rightarrow$ $3A = 1$

$A = 1/3$ $B = -1/3$

$\boxed{2}$

need to flip it over to integrate with respect to x.

10 (ii) (a) $\frac{dx}{dt} = K(3-x)(6-x)$

$\frac{dt}{dx} = \frac{1}{K(3-x)(6-x)}$

$\frac{dt}{dx} = \frac{1}{K}\left(\frac{1/3}{3-x} - \frac{1/3}{6-x}\right)$

Integrate $\longrightarrow$

$t = \frac{1}{3K}\int \frac{1}{3-x} - \frac{1}{6-x}\,dx$ easier to integrate if you take constants $\frac{1}{K}$ and $\frac{1}{3}$ out.

$t = \frac{1}{3K}(-\ln|3-x| + \ln|6-x|) + c$

* Remember log laws $\log a - \log b = \log\left(\frac{a}{b}\right)$

sub in $t=0$, $x=0$

$0 = \frac{1}{3K}(-\ln 3 + \ln 6) + c$

$0 = \frac{1}{3K}\left(\ln\left(\frac{6}{3}\right)\right) + c$

$c = -\frac{1}{3K}\ln 2$

sub in $t=1$, $x=1$, $c=-\frac{1}{3K}\ln 2$

$1 = \frac{1}{3K}(-\ln 2 + \ln 5) - \frac{1}{3K}\ln 2 \quad (\times 3K)$

$3K = \ln\left(\frac{5/2}{2}\right)$

$K = \frac{1}{3}\ln\frac{5}{4}$

$\boxed{7}$

(b) sub $t=2$, $K = \frac{1}{3}\ln\frac{5}{4}$ and $c = -\frac{1}{3K}\ln 2$

$2 = \frac{1}{3(\frac{1}{3}\ln\frac{5}{4})}\left(-\ln|3-x| + \ln|6-x|\right) - \frac{1}{3(\frac{1}{3}\ln\frac{5}{4})}\ln 2$

$(\times \ln\frac{5}{4})$

$2\ln\frac{5}{4} = -\ln|3-x| + \ln|6-x| - \ln 2$

(use log laws)

$\ln\left(\frac{5}{4}\right)^2 = \ln\left|\frac{6-x}{2(3-x)}\right|$

(e both sides)

$\frac{25}{16} = \frac{6-x}{2(3-x)}$

$25 \times 2(3-x) = 16(6-x)$

$150 - 50x = 96 - 16x$

$54 = 34x$

$x = \frac{27}{17}$

$\boxed{4}$