June 2016 C4

1. $$\begin{array}{r} 2x+4 \\ 2x^2+0x+1 \,\big)\, 4x^3+8x^2-5x+12 \\ -\left(4x^3+0+2x\right) \\ 8x^2-7x+12 \\ -\left(8x^2+0x+4\right) \\ -7x+8 \end{array}$$

quotient $2x+4$ remainder $-7x+8$

2. $\int_{\pi/16}^{\pi/8} (9-6\cos^2 4x)\,dx$

$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$

$\Rightarrow 2\cos^2 x = \cos 2x + 1$ $\quad 6\cos^2 x = 3\cos 2x + 3$ and $6\cos^2 4x = 3\cos 8x + 3$

so $\int_{\pi/16}^{\pi/8} 9 - 3\cos 8x - 3\,dx = \int_{\pi/16}^{\pi/8} 6 - 3\cos 8x\,dx$

$= \left[6x - \frac{3\sin 8x}{8}\right]_{\pi/16}^{\pi/8} = \left(\frac{6\pi}{8}\right) - \left(\frac{6\pi}{16} - \frac{3}{8}\right)$

$= \frac{6\pi}{16} + \frac{3}{8} = \frac{3}{8} \; \frac{+3\pi}{8}$

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3. $y\sin 2x + x^{-1} + y^2 = 5$

$$\frac{dy}{dx}\sin 2x + 2y\cos 2x - x^{-2} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(\sin 2x + 2y\right) = \frac{1}{x^2} - 2y\cos 2x$$

$$\frac{dy}{dx} = \frac{\frac{1}{x^2} - 2y\cos 2x}{\sin 2x + 2y}$$

4. $$\int_1^8 x^{-1/3}\ln x\,dx = \left[\frac{3}{2}x^{2/3}\ln x\right] - \int \frac{3}{2}x^{-1/3}\,dx$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

let $\frac{dv}{dx} = x^{-1/3}$ $\qquad u = \ln x$

$v = \frac{3}{2}x^{2/3}$ $\qquad \frac{du}{dx} = 1/x$

$$= \left[\frac{3}{2}x^{2/3}\ln x - \frac{9}{4}x^{2/3}\right]_1^8 = 6\ln 8 - 9 + \frac{9}{4} = 6\ln 8 - \frac{27}{4}$$

$$= 6\ln 2^3 - \frac{27}{4} = 18\ln 2 + \frac{27}{4}$$

3

5.(i) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + t\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$

so
$$1 + 2s = 3 + 5t \qquad\qquad 2s - 5t = 2 \quad (1)$$
$$4 - s = 2 - 3t \quad \Rightarrow \quad s - 3t = 2 \quad (2)$$
$$5 + 3s = -5 + t \qquad\qquad 3s - t = -10 \quad (3)$$

$(1) - 5(2) \Rightarrow$
$$\begin{array}{r} 2s - 5t = 2 \\ 15s - 5t = -50 \\ \hline -13s \quad = 52 \end{array} \Rightarrow s = -4$$

$-4 - 3t = 2 \Rightarrow t = -2$

so $x = 1 - 4\times 2 = -7 \quad y = 4 + -4\times -1 = 8 \quad z = 5 - 4\times 3 = -7$

$(-7, 8, -7)$ point of intersection

(ii) $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times 2 = \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$ so direction vectors are parallel

$\begin{pmatrix} 7 \\ 1 \\ 14 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} = s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow s = 3$

so $(7, 1, 14)$ on L and $\begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$ along L $\therefore$

L can be represented by $\underset{\sim}{r}$.

(4)

(6) $\int \frac{6x^3+4x}{\sqrt{x^2-2}}\,dx$

let $u = x^2-2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$x^2 = u+2$

$\int \frac{2x(3x^2+2)}{u^{1/2}} \quad \frac{du}{2x}$

so $\int \frac{3(u+2)+2}{u^{1/2}}\,du = \int \frac{3u+8}{u^{1/2}}\,du$

$= \int 3u^{1/2} + 8u^{-1/2}\,du = \left[\frac{6}{3}u^{3/2} + 16u^{1/2} + k\right]$

$2(x^2-2)^{3/2} + 16(x^2-2)^{1/2} + K$

(5)

7. $(1+kx)^n \approx 1 - 6x + 30x^2 + \cdots$

$$nk = -6 \Rightarrow k = \frac{-6}{n}$$

$$\frac{n(n-1)K^2}{2} = 30 \quad \Rightarrow n(n-1)K^2 = 60$$

so sub k in $\quad n(n-1)\left(\frac{-6}{n}\right)^2 = 60$

$$(n^2 - n)\,\frac{36}{n^2} = 60$$

$$36n^2 - 36n = 60n^2$$

$$0 = 24n^2 + 36n = 12n(2n+3)$$

$\Rightarrow \cancel{n = 0}$ or $n = -3/2$

$\cancel{K = \infty}$ $\quad K = \frac{-6}{-3/2} \Rightarrow 4$ $\quad$ so $\underline{\underline{n = -3/2}}$ $\quad \underline{\underline{K = 4}}$

⑥ a. $\vec{OA} \cdot \vec{OB} = 0$

so $3\sin\alpha \times 2\cos\alpha + 2\cos\alpha \times 4\sin\alpha - 3 = 0$

$6\sin\alpha\cos\alpha + 8\sin\alpha\cos\alpha - 3 = 0$

$14\sin\alpha\cos\alpha = 3$

$\sin\alpha\cos\alpha = \frac{3}{14}$

but $\sin(A+B) = \sin A\cos B + \cos A\sin B$

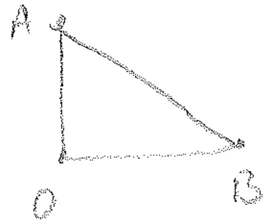
so $\sin(\alpha+\alpha) = 2\sin\alpha\cos\alpha$

$\therefore \quad \frac{1}{2}\sin(2\alpha) = \frac{3}{14} \Rightarrow \quad \sin 2\alpha = 3/7$

$2\alpha = \sin^{-1}(3/7) = 0.442911044\,1$ or 2.69868161

so $2\alpha = 25.37693353$ or 154.6230665

$\alpha = 12.68846677$ $\quad 77.31153325$



Area $= \frac{1}{2} \times OA \times OB$

$= \frac{1}{2} \times \sqrt{3^2\sin^2\alpha + 2^2\cos^2\alpha + (-1)^2}$

$\times \sqrt{2^2\cos^2\alpha + 4^2\sin^2\alpha + 3^2}$

$= \frac{1}{2} \times 2.28937333 \times 3.684963037$

$= 4.218128049$

7

9. $x = 1 - \cos t$ $\quad 0 \le t \le \pi$

$y = \sin t \sin 2t$

(i) at x axis $y = 0$

$0 = \sin t \sin 2t$

$t = 0, \pi/2, \pi$ $\quad x = 1 - \cos 0 = 0 \quad (0,0)$

$x = 1 - \cos \pi/2 = 1 \quad (1,0)$

$x = 1 - \cos \pi = 2 \quad (2,0)$

(ii) $\frac{dx}{dt} = \sin t$ $\quad \frac{dy}{dt} = \cos t \sin 2t + 2 \sin t \cos 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cos t \sin 2t + 2 \sin t \cos 2t}{\sin t}$$

but $\sin 2t = 2 \sin t \cos t$

so $\frac{dy}{dx} = \frac{2\cos^2 t \cancel{\sin t} + 2 \cos 2t \cancel{\sin t}}{\cancel{\sin t}} = 2 \cos 2t + 2 \cos^2 t$

$\frac{dy}{dx} = 0 \quad 0 = 2\cos 2t + 2\cos^2 t \quad \Rightarrow$

but $\cos 2t = \cos^2 t - \sin^2 t$

$0 = 2\cos^2 t - 2 \sin^2 t + 2\cos^2 t$

but $\sin^2 t + \cos^2 t \equiv 1 \quad \Rightarrow \sin^2 t \equiv 1 - \cos^2 t$

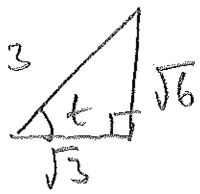
$0 = 4\cos^2 t - 2 + 2\cos^2 t \quad \Rightarrow \quad 2 = 6 \cos^2 t$

$\Rightarrow \quad 1/3 = \cos^2 t \quad \Rightarrow \frac{\pm\sqrt{3}}{3} = \cos t \quad \Rightarrow x = 1 - \frac{\sqrt{3}}{3} = \frac{3 - \sqrt{3}}{3}$

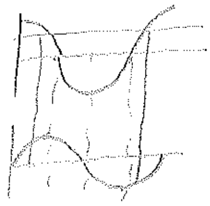
or $x = 1 + \frac{\sqrt{3}}{3} = \frac{3 + \sqrt{3}}{3}$

(8) 9.

$\cos t = \frac{\sqrt{3}}{3}$



$\therefore \sin t = \frac{\sqrt{6}}{3}$



$\sin 2t = 2\times\frac{\sqrt{6}}{3}\times\frac{\sqrt{3}}{3} = \frac{6\sqrt{2}}{9} = \frac{2\sqrt{2}}{3}$

$\sin t \sin 2t = \frac{\sqrt{6}}{3}\times\frac{2\sqrt{2}}{3} = \frac{4\sqrt{3}}{9}$

$\left(\frac{3-\sqrt{3}}{3}, \frac{4\sqrt{3}}{9}\right) \quad \left(\frac{3+\sqrt{3}}{3}, \frac{4\sqrt{3}}{9}\right)$

(iii) $x = 1-\cos t$

so $\cos t = 1-x$

$y = \sin t \sin 2t = 2\sin^2 t \cos t$

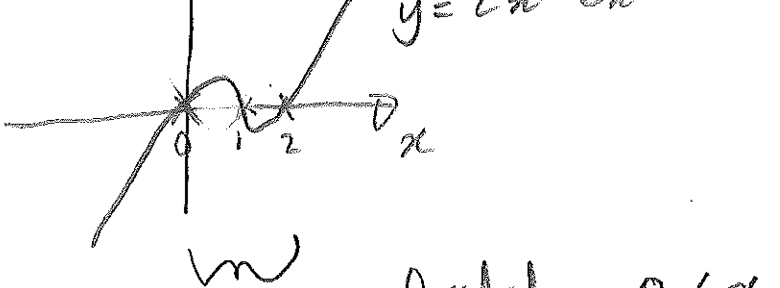
$y = 2(1-x)\left(1-(1-x)^2\right)$

$= 2(1-x)\left(1-(1-2x+x^2)\right)$

$= (2-2x)(2x-x^2) = 4x - 2x^2 - 4x^2 + 2x^3$

$\underline{\underline{y = 2x^3 - 6x^2 + 4x}}$

(iv)



but curve limited $0 \le x \le 2$ as t between 0 and π

(9) 10. (i) $\dfrac{-2x^2+5x+16}{(x+1)^2(x+4)}$

let $\dfrac{-2x^2+5x+16}{(x+1)^2(x+4)} \equiv \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+4)}$

$\Rightarrow \times (x+1)^2 (x+4)$

$-2x^2+5x+16 \equiv A(x+1)(x+4)+B(x+4) + C(x+1)^2$

let $x = -1$

$-2(-1)^2+5(-1)+16 = B(-1+4)$

$-2-5+16 \quad = 3B \quad \Rightarrow \quad \underline{\underline{B=3}}$

let $x = -4$ $\quad -2(-4)^2+5(-4)+16 = C(-4+1)^2$

$-32 \quad -20 \quad +16 = 9C$

$-36 = 9C \Rightarrow C = \underline{\underline{-4}}$

let $x=0$ $\quad 16 = 4A + 12 - 4$

$A=2$

So $\quad \dfrac{2}{x+1} + \dfrac{3}{(x+1)^2} - \dfrac{4}{(x+4)}$

10

10. $\frac{dy}{dx} = \left(\frac{2}{x+1} + \frac{3}{(x+1)^2} - \frac{4}{x+4} \right) y$

so $\frac{1}{y} \frac{dy}{dx} = \frac{2}{x+1} + \frac{3}{(x+1)^2} - \frac{4}{x+4}$

$\ln|y| = 2 \ln|x+1| - 3 (x+1)^{-1} - 4 \ln|x+4| + k$

$\ln|y| = \ln\left| \frac{(x+1)^2}{(x+4)^4} \right| - \frac{3}{x+1} + k$

$y = \left(\frac{(x+1)^2}{(x+4)^4} \right) e^{-3/x+1} C$

when $x = 0$ $y = \frac{1}{256}$

$y = \frac{1}{256} e^{-3} C \Rightarrow C = e^3$

so $y = \left(\frac{(x+1)^2}{(x+4)^4} \right) e^{-3/x+1} e^3$

if $x = 2$ $y = \frac{3^2}{6^4} e^{-1} e^3 = \frac{1}{144} e^2$