Core 4 - June 2007 - Miss Watson's rough solutions

(1) (i) $\dfrac{3x+1}{(x+2)(x-3)} = \dfrac{A}{(x+2)} + \dfrac{B}{(x-3)}$

$A(x-3) + B(x+2) = 3x+1$

$\boxed{Ax} - 3A + \boxed{Bx} + 2B = 3x+1$

$-3A + 2B = 1$

$A + B = 3 \quad (\times 2)$

$2A + 2B = 6$

$-5A = -5$

$A = 1$

$B = 2$

$\dfrac{1}{(x+2)} + \dfrac{2}{(x-3)}$ [2]

(ii) $f(x) = (x+2)^{-1} + 2(x-3)^{-1}$

$f'(x) = -(x+2)^{-2} - 2(x-3)^{-2}$

$= \dfrac{-1}{(x+2)^2} - \dfrac{2}{(x+3)^2}$

always positive because squared

so both terms will stay negative. [3]

2 $\int_0^1 x^2 e^x$

$u = x^2 \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2x \quad v = e^x$

$\left[x^2 e^x\right]_0^1 - \int 2x e^x \, dx$ $\quad u = 2x \quad \frac{du}{dx} = 2 \quad \frac{dv}{dx} = e^x \quad v = e^x$

$\left[x^2 e^x\right]_0^1 - \left(\left[2x e^x\right]_0^1 - \int 2e^x \, dx\right)$

$\left[x^2 e^x\right]_0^1 - \left(\left[2x e^x\right]_0^1 - \left[2e^x\right]_0^1\right)$

$\left(e^1 - 0\right) - \left(\left(\cancel{2e^1} - 0\right) - \left(\cancel{2e^1} - 2\right)\right)$

$e^1 - (2)$

$\underline{\underline{e^1 - 2}}$

$\boxed{6}$

3

$$\pi \int_0^{\pi} r^2 \, dx$$

$$\pi \int_0^{\pi} \sin^2 x \, dx$$

$\cos 2A = \cos^2 A - \sin^2 A$

$(\cos^2 A = 1 - \sin^2 A)$

$$\pi \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$$

$\cos 2A = 1 - \sin^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$

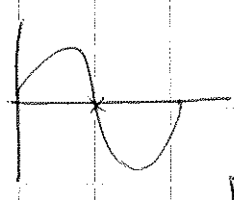
$2\sin^2 A = 1 - \cos 2A$

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$

$$\pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x \right]_0^{\pi}$$

$$\pi \left(\left(\frac{\pi}{2} - 0 \right) - \left(0 - 0 \right) \right)$$

$$\frac{\pi^2}{2}$$



$\boxed{6}$

4 (i) $(2+x)^{-2}$ up to x^3

$2^{-2}\left(1+\frac{1}{2}x\right)^{-2}$

$\frac{1}{4}\left(1+\frac{1}{2}x\right)^{-2}$ $\quad n=-2$, $x \to \frac{1}{2}x$

$$\frac{1}{4}\left(1+(-2)\left(\frac{1}{2}x\right)+\frac{(-2)(-3)}{2}\left(\frac{1}{2}x\right)^2+\frac{(-2)(-3)(-4)}{2\times 3}\left(\frac{1}{2}x\right)^3\right)$$

$$\frac{1}{4}\left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3\right)$$

$$\frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 - \frac{1}{8}x^3$$

~~$|x|<1$~~

$\frac{1}{2}x < 1$

$|x| < 2$

$\boxed{5}$

(ii) ~~$1+x^2 = \ldots$~~

$(1+x^2)(2+x)^{-2}$

	$\frac{1}{4}$	$-\frac{1}{4}x$	$+\frac{3}{16}x^2$	$-\frac{1}{8}x^3$
1	$\frac{1}{4}$	$-\frac{1}{4}x$	$+\frac{3}{16}x^2$	$-\frac{1}{8}x^3$
x^2	$\frac{1}{4}x^2$	$-\frac{1}{4}x^3$		

$$-\frac{1}{8}x^3 - \frac{1}{4}x^3 = -\frac{3}{8}x^3$$

coeff $= -3/8$

$\boxed{2}$

(5) $x = \cos t \quad y = 3 + 2\cos 2t$

(i) $\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = -4\sin 2t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{-}4\sin 2t}{\cancel{-}\sin t}$$

double angle
$\sin 2t = 2\sin t\cos t$

$$= \frac{8\cancel{\sin t}\cos t}{\cancel{\sin t}}$$

$$= 8\cos t$$

max value of $\cos t = 1$
so max $= 8 \times 1$
$= \underline{\underline{8}}$

$\boxed{4}$

(ii) $x = \cos t \qquad y = 3 + 2\cos 2t$

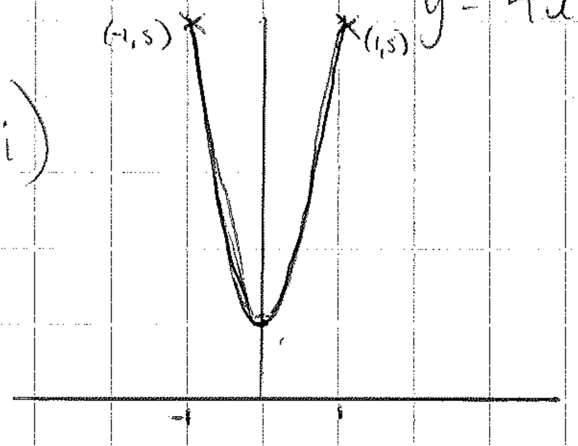
$\cos 2A = \cos^2 A - \sin^2 A$
$(1 - \cos^2 A)$
$\cos 2A = \cos^2 A - 1 + \cos^2 A$
$\cos 2A = 2\cos^2 A - 1$

$y = 3 + 4\cos^2 t - 2$
$y = 1 + 4\cos^2 t$

$y = 1 + 4x^2$
$y = 4x^2 + 1$

$\boxed{3}$

(iii)



$\boxed{2}$

6) $x^2 + 3xy + 4y^2 = 58$

$2x + 3x\frac{dy}{dx} + 3y + 8y\frac{dy}{dx} = 0$

$3x\frac{dy}{dx} + 8y\frac{dy}{dx} = -3y - 2x$ $\quad u = 3x \quad v = y$

$\frac{du}{dx} = 3 \quad \frac{dv}{dx} = 1\frac{dy}{dx}$

$\frac{dy}{dx}(3x + 8y) = -3y - 2x$

$\frac{dy}{dx} = \frac{-3y - 2x}{3x + 8y}$ ← (2, 3)

$= \frac{-\cancel{24}\,9 - 4}{6 + 24} = \frac{-13}{30}$

$\cancel{= \frac{-28}{30} = \frac{-14}{15}}$

grad normal $= \frac{30}{13}$

$y = \frac{30}{13}x + c$ $\quad (2, 3)$

$3 = \frac{60}{13} + c$

$c = -1\frac{8}{13} = \frac{-21}{13}$

$y = \frac{30}{13}x - \frac{21}{13}$ $\quad \times(13)$

$13y = 30x - 21$ $\quad \boxed{8}$

$13y - 30x + 21 = 0$ $\quad 30x - 13y - 21 = 0$

7

(i)

$$\begin{array}{r|l} & 2x + 3 \\ \hline x^2+4 & 2x^3 + 3x^2 + 9x + 12 \\ & -\;2x^3 \qquad\quad + 8x \\ & \qquad 3x^2 + x + 12 \\ & -\;3x^2 \qquad\quad + 12 \\ & \qquad\qquad\quad \underline{\underline{x}} \end{array}$$

quotient = $2x+3$
remainder = x

$\boxed{4}$

(ii) $2x + 3 + \dfrac{x}{x^2+4}$

$A = 2$
$B = 3$
$C = 1$
$D = 0$

$\boxed{1}$

(iii) $\displaystyle\int_1^3 2x + 3 + \frac{x}{x^2+4}\, dx$

$$\left[x^2 + 3x + \tfrac{1}{2}\ln|x^2+4| \right]_1^3$$

$$\left(9 + 9 + \tfrac{1}{2}\ln 13\right) - \left(1 + 3 + \tfrac{1}{2}\ln|5|\right)$$

$14 + \frac{1}{2}\ln 13 - \frac{1}{2}\ln 5$

$\underline{14 + \frac{1}{2}\ln\left(\frac{13}{5}\right)}$

$\boxed{5}$

8 $\quad \dfrac{dh}{dt} = \dfrac{6-h}{60}$

(i) $\displaystyle\int \frac{dt}{dh}\,dh = \int \frac{20}{6-h}\,dh$

$t = -20\ln|6-h| + c \qquad \begin{matrix} t=0 \\ h=1 \end{matrix}$

$0 = -20\ln|5| + c$

$t = -20\ln|6-h| + 20\ln|5|$

$t = 20\left(\ln|5| - \ln|6-h|\right)$

$t = 20\ln\left(\dfrac{5}{6-h}\right)$ $\quad \boxed{6}$

(ii) $t = 20\ln\left(\dfrac{5}{4}\right)$

$t = 4.46$ years $\quad \boxed{1}$

(iii) $10 = 20\ln\left(\dfrac{5}{6-h}\right)$

$\tfrac{1}{2} = \ln\left(\dfrac{5}{6-h}\right)$

$e^{1/2} = \dfrac{5}{6-h}$

$6e^{1/2} - he^{1/2} - 5 = 0$

$\dfrac{6e^{1/2} - 5}{e^{1/2}} = h$

$\underline{h = 2.967 \text{ metres}}$ $\quad \boxed{2}$

(iv) ln (pos number)

so $h < 6$

approx $\underline{\underline{6\text{ m}}}$ $\quad \boxed{1}$

(9) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} + s \begin{pmatrix} -6 \\ 8 \\ -2 \end{pmatrix}$ $L1$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ $L2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + u \begin{pmatrix} 3 \\ c \\ 1 \end{pmatrix}$ $L3$

(i) $a = \begin{pmatrix} -6 \\ 8 \\ -2 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ $\cos\theta = \dfrac{a \cdot b}{|a||b|}$

$a \cdot b = 14$ $\cos\theta = \dfrac{14}{\sqrt{104}\sqrt{14}} = \dfrac{14}{\sqrt{1456}}$

$|a| = \sqrt{104}$

$|b| = \sqrt{14}$

$\theta = \cos^{-1}\left(\dfrac{14}{\sqrt{1456}}\right)$

$\cancel{\theta = 89.449}°$

$\theta = 68.4754 \ldots$

$\underline{68.48}$ degrees

$= 1.1951$

$\underline{1.2}$ radians $\boxed{4}$

(ii) $\begin{pmatrix} -6 \\ 8 \\ -2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ c \\ 1 \end{pmatrix}$ $\begin{pmatrix} 6 \\ -8 \\ 2 \end{pmatrix}$ parallel if scalar products of each other

$\underline{c = -4}$ $\boxed{2}$

(iii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ $L2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + u\begin{pmatrix} 3 \\ c \\ 1 \end{pmatrix}$ $L3$

$x = 3 + t$ $\quad y = -8 + 3t$ $\quad z = 2t$

$x = 2 + 3u$ $\quad y = 1 + cu$ $\quad z = 3 + u$

$3 + t = 2 + 3u$ $\qquad 2t = 3 + u$

$t - 3u = -1$ $\times 2$ $\qquad 2t - u = 3$

$2t - u = 3$

$-\ 2t - 6u = -2$

$5u = 5$

$\underline{\underline{u = 1}}$

$t - 3 = -1$

$\underline{\underline{t = 2}}$

$L2$ $\begin{pmatrix} 3 + 2 \\ -8 + 6 \\ 0 + 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$

$L3$ $\begin{pmatrix} 2 + 3 \\ 1 + c \\ 3 + 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 + c \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$ so $\underline{\underline{c = -3}}$

$\boxed{5}$